Math 150
Final Exam
Professor Busken

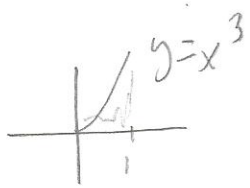
Name: Key

1. (5 points) Evaluate $\lim\limits_{x\to 5}\left(\dfrac{x-\frac{10}{x}-3}{x-5}\right)$.

$$= \lim_{x\to 5}\left[\frac{\frac{x^2-3x-10}{x}}{x-5}\right] = \lim_{x\to 5}\left[\frac{1}{\cancel{x-5}}\cdot\frac{\cancel{(x-5)}(x+2)}{x}\right] = \lim_{x\to 5}\left(\frac{x+2}{x}\right)$$

$$= \frac{7}{5}$$

2. (5 points) If the arc of the function $f(x) = x^3$ for $x \in [0,1]$ is revolved about the x-axis, find the area of the surface generated. (Distance in meters)



$$SA = 2\pi\int_0^1 x^3\sqrt{1+(3x^2)^2}\,dx$$

$$= 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx = \frac{2\pi}{36}\int_1^{10} u^{1/2}\,du$$

Let $u = 1+9x^4$

$du = 36x^3\,dx$

and $\dfrac{du}{36} = x^3\,dx$

$$\frac{0 \longrightarrow 1}{x} \qquad \frac{1 \longrightarrow 10}{u}$$

$$= \frac{\pi}{18}\,\frac{2}{3}\left(u^{3/2}\right)\Big|_{u=1}^{u=10} = \frac{\pi}{27}\left(10^{3/2}-1\right)$$

$$\doteq \boxed{3.563\ m^2}$$

Key

$\frac{d}{dx}\left(x^{-1/2}\right) = -\frac{1}{2}x^{-3/2} = \frac{-1}{2x\sqrt{x}}$ #4

$\int_1^4 x^{-1/2}dx = 2x^{1/2}\Big|_1^4 = 2(\sqrt{4}-\sqrt{1}) = 2(2-1) = 2$

3. (5 points) Evaluate $\lim_{\theta\to 0}\left(\frac{\sin^5(2\theta)}{16\theta^5}\right)$.

$= 2\lim_{\theta\to 0}\left(\frac{\sin 2\theta}{2\theta}\right)^5 = 2\left(\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}\right)^5 = 2\cdot 1^5$

$= 2$

4. (5 points) Verify the Mean Value Theorem (for Integrals) for $f(x) = \frac{1}{\sqrt{x}}$ on $[1,4]$.

f is cont on [1,4] since compositions of continuous functions are continuous. Thus, By the MVT $\exists$ a number $x=c$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$, or equivalently, that

$\left[\frac{1}{\sqrt{c}} = \frac{1}{3}\int_1^4 x^{-1/2}dx\right] \Leftrightarrow \left[\frac{1}{\sqrt{c}} = \frac{1}{3}\cdot 2\right] \Leftrightarrow \left[\sqrt{c} = \frac{3}{2}\right]$

$\Leftrightarrow \left[c = \left(\frac{3}{2}\right)^2\right] \Leftrightarrow \left[c = 9/4\right].$

5. (5 points) Find $\int \frac{\sin(x)}{\cos(x)+2}\,dx$

let $u = \cos(x)+2$

then $du = -\sin x\,dx$

or $-du = \sin x\,dx$

$= -\int \frac{1}{u}du = -\ln|u| + C$

$= -\ln|\cos(x)+2| + C$

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6. (5 points) Find $\int \frac{4^x}{1+16^x}\,dx = \int \frac{4^x}{1+(4^x)^2}\,dx$

Let $u = 4^x$

$du = 4^x \ln 4 \, dx$

$\frac{du}{\ln 4} = 4^x\,dx$

$$= \frac{1}{\ln 4}\int \frac{1}{1+u^2}\,du$$

$$= \boxed{\frac{1}{\ln 4}\tan^{-1}(4^x) + C}$$

7. (5 points) Find $\int \frac{1}{x\sqrt{16-\ln^2(x)}}\,dx$

$$= \int \frac{1}{x\sqrt{16\left[1-\left(\frac{\ln x}{4}\right)^2\right]}}\,dx = \frac{1}{4}\int \frac{1}{x\sqrt{1-\left(\frac{\ln x}{4}\right)^2}}\,dx$$

Let $u = \frac{1}{4}\ln x$

then $du = \frac{1}{4x}\,dx$

$$= \int \frac{1}{\sqrt{1-\left(\frac{\ln x}{4}\right)^2}}\cdot\frac{1}{4x}\,dx$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}(u) + C$$

$$= \sin^{-1}\left(\frac{1}{4}\ln x\right) + C$$

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8. (5 points) Find $\int \frac{1}{x\sqrt{x^6-49}}\,dx$ $= \int \frac{1}{x\sqrt{49\left(\frac{x^6}{49}-1\right)}}\,dx$

$= \frac{1}{7}\int \frac{1}{x\sqrt{\left(\frac{x^3}{7}\right)^2-1}}\,dx = \int \frac{1}{x^3\sqrt{\left(\frac{x^3}{7}\right)^2-1}}\,\frac{x^2}{7}\,dx$

let $u = \frac{x^3}{7}$

then $du = \frac{3x^2}{7}\,dx$

or $\frac{du}{3} = \frac{x^2}{7}\,dx$

$7u = x^3$

$= \int \frac{1}{7u\sqrt{u^2-1}}\,\frac{du}{3} = \frac{1}{21}\int \frac{1}{u\sqrt{u^2-1}}\,du$

$= \frac{1}{21}\sec^{-1}(u) + C$

$= \boxed{\frac{1}{21}\sec^{-1}\left(\frac{x^3}{7}\right) + C}$

9. (5 points) Find $\int \frac{e^{2x}}{\sin(e^{2x})}\,dx$ $= \frac{1}{2}\int \frac{1}{\sin(u)}\,du$

let $u = e^{2x}$

then $du = 2e^{2x}\,dx$

$\frac{du}{2} = e^{2x}\,dx$

$= \frac{1}{2}\int \csc(u)\,du$

$= \frac{1}{2}\ln\left|\csc(u) - \cot(u)\right| + C$

$= \boxed{\frac{1}{2}\ln\left|\csc(e^{2x}) - \cot(e^{2x})\right| + C}$

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Let $u = x^{1/2}$ then $du = \frac{1}{2\sqrt{x}}\,dx$

10. (5 points) Find $\int \frac{1}{\sqrt{x}\cos(\sqrt{x})}\,dx$ $= 2\int \frac{1}{\cos(u)}\,du$

$= 2\int \sec(u)\,du = 2\ln|\sec(u) + \tan(u)| + c$

$= \boxed{2\ln|\sec(\sqrt{x}) + \tan(\sqrt{x})| + c}$

11. (5 points) Find $\frac{dy}{dx}$ if $y = \ln\left(x + \tan(x)\right)$

$\frac{dy}{dx} = \frac{1}{x+\tan(x)} \cdot \frac{d}{dx}\left(x + \tan(x)\right) = \boxed{\frac{1+\sec^2(x)}{x+\tan(x)}}$

$=$

12. (5 points) Find $\frac{dy}{dx}$ if $y = e^{3x}\sin(4x)$

$\frac{dy}{dx} = 3e^{3x}\sin(4x) + 4e^{3x}\cos(4x)$

$= \boxed{e^{3x}\left(3\sin(4x) + 4\cos(4x)\right)}$

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13. (5 points) Let $f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & \text{if } x \neq 4 \\ A & \text{if } x = 4 \end{cases}$. Determine the value of A which makes f continuous at $x = 4$.

$A = 8$

14. (5 points) If $x^3 + \cos(xy^2) + y^4 = 3y - 2x - 1$, find y'. Assume $y = f(x)$.

$$\frac{d}{dx}\left(x^3 + \cos(xy^2) + y^4\right) = \frac{d}{dx}(3y - 2x - 1)$$

$$3x^2 - \sin(xy^2) \cdot \frac{d}{dx}(xy^2) + 4y^3y' = 3y' - 2$$

$$3x^2 - \sin xy^2 \left[y^2 + 2xy\,y'\right] + 4y^3y' = 3y' - 2$$

$$3x^2 - y^2\sin(xy^2) - 2xy\,y'\sin(xy^2) = 3y' - 2 - 4y^3y'$$

$$3x^2 - y^2\sin(xy^2) + 2 = y'\left(2xy\sin(xy^2) + 3 - 4y^3\right)$$

$$\boxed{y' = \frac{3x^2 - y^2\sin(xy^2) + 2}{2xy\sin(xy^2) + 3 - 4y^3}}$$

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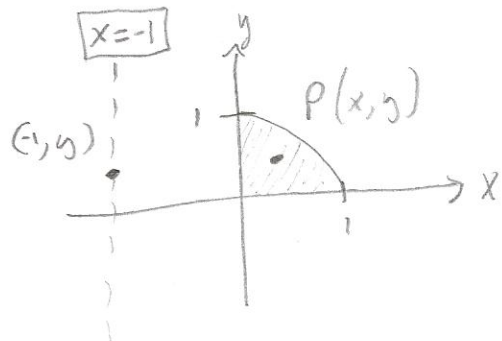
Extra Credit Problem

15. (5 points) A lamina (with uniform thickness 0.001 m) has the shape of the region Ω in the xy-plane bounded by the graphs of $y = 1 - x^2$ and $y = 0$, between $x = 0$ and $x = 1$. If the density (in kg/m^3) at each point P is inversely proportional to the distance from P to the line $x = -1$ with $\delta(0.5, 0, 0) = 4/3\ kg/m^3$. Find the mass of the lamina.

Density function

$$\delta = \frac{k}{\sqrt{(x+1)^2 + (y-y)^2}} = \frac{k}{x+1}$$



or $\boxed{\delta = \frac{k}{x+1}}$ then $\delta(0.5,0,0) = \frac{4}{3}$

$$\Rightarrow \left[\frac{4}{3} = \frac{k}{0.5+1}\right] \Leftrightarrow \left[\frac{4}{3} = \frac{k}{\frac{3}{2}}\right] \Leftrightarrow \left[\frac{3}{2}\cdot\frac{4}{3} = k\right] \Leftrightarrow \left[k = 2\right]$$

$$\Rightarrow \boxed{\delta = \frac{2}{x+1}}$$

$$\Rightarrow \text{mass} = (\text{volume})(\text{density})$$
$$= (\text{thickness})\cdot(\text{area of lamina})\cdot(\text{density})$$
$$= 0.001 \int_0^1 (1-x^2)\cdot\left(\frac{2}{x+1}\right)dx = 0.001 \int_0^1 (1-x)\cancel{(1+x)}\cdot\frac{2}{\cancel{x+1}}\,dx$$
$$= 0.001 \int_0^1 2(1-x)\,dx = 0.002 \int_0^1 (1-x)\,dx$$
$$= 0.002\left[x - \frac{1}{2}x^2\right]_0^1 = 0.002\left(1 - \frac{1}{2}\right) = 0.002 \cdot \frac{1}{2}$$
$$= \boxed{0.001\ \text{kg}}$$