Monday 3/2

Test 2 - Thurs., March 12^{th}

Over Sections 1.7, 4.1, 3.1, 3.2, 3.3

Quiz 5 - due Tuesday, 3/3

Quiz 6 - due by 12 pm, 3/6.

Test 2 Notes:

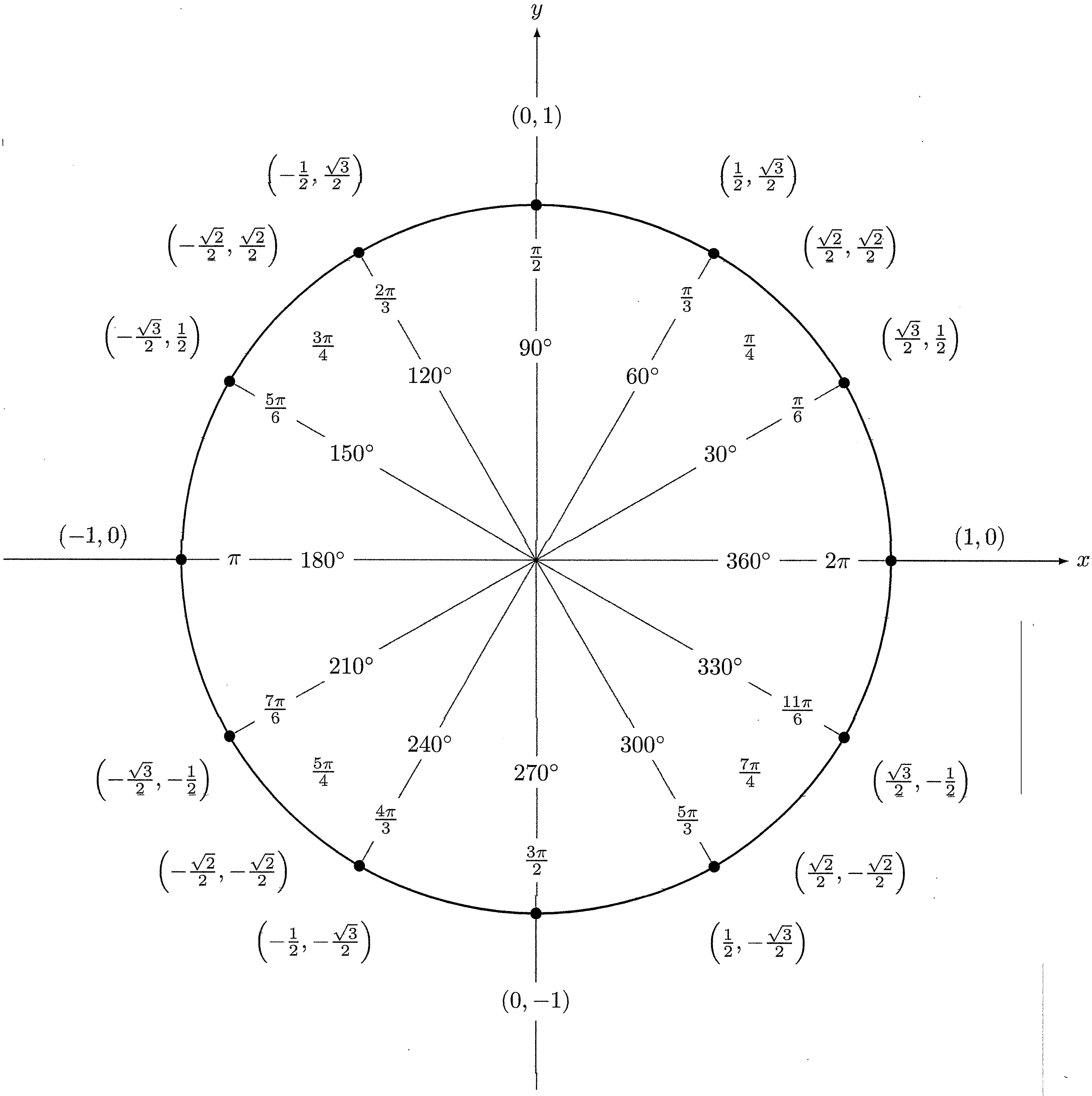
① Know the properties of inverse functions

② Know SOH-CAH-TOA (the defns of sine, cosine and tangent in terms of the ratios of the side lengths of a right triangle having acute angle, θ)

③ Know the domains & ranges of $y = \sin^{-1}(x)$, $y = \cos^{-1}(x)$, and $y = \tan^{-1}(x)$.

↑ ↑ ↑

These things must be memorized. You need to be able to recall the image of the unit circle along with the common angles in the interval $[0, 2\pi)$ and their sines and cosines.



y
x
$(0, 1)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$(1, 0)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$(0, -1)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$(-1, 0)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{6}$ 30°
$\frac{\pi}{4}$
$\frac{\pi}{3}$ 60°
$\frac{\pi}{2}$ 90°
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$
$\frac{5\pi}{6}$ 150°
π 180°
$\frac{7\pi}{6}$ 210°
$\frac{5\pi}{4}$
$\frac{4\pi}{3}$ 240°
$\frac{3\pi}{2}$ 270°
$\frac{5\pi}{3}$ 300°
$\frac{7\pi}{4}$
$\frac{11\pi}{6}$ 330°
360° 2π

① Properties of inverse functions.

* The only functions that have inverses are one to one functions. One-to-one functions have graphs that pass the horizontal line test. We say they pass the test because you can sweep a horizontal line across the graph and every horizontal line intersects the function graph in at most one point. One-to-one functions never have any y-value used twice. This is so we can define an inverse function from the range over to the domain.

* If f is one-to-one, then f^{-1} exists, and it's unique (there is only one inverse function, f^{-1})

* $\mathrm{dom}(f^{-1}) = \mathrm{rng}(f)$

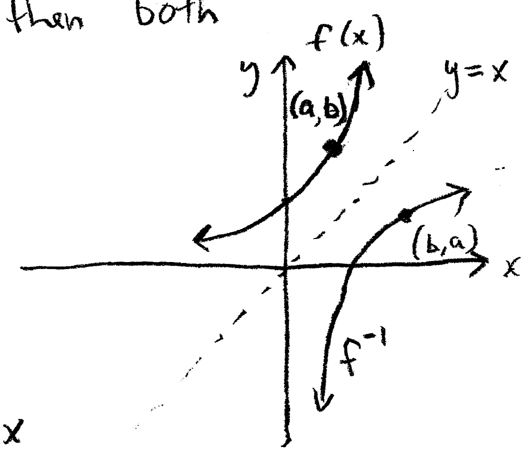
* $\mathrm{rng}(f^{-1}) = \mathrm{dom}(f)$

* $[\,y = f(x)\,] \Leftrightarrow [\,f^{-1}(y) = x\,]$

* If x is in the domain of f, then both

① $f(f^{-1}(x)) = x$ and

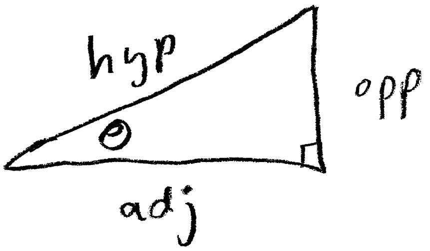
② $f^{-1}(f(x)) = x$.



* $\boxed{\text{Symmetry Property}}$ The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric to the line $y=x$

(2) Defns of Sine, Cosine and Tangent as they relate to a right triangle having acute angle, θ.

SOH-CAH-TOA



$$\sin\theta = \frac{opp}{hyp}, \quad \cos\theta = \frac{adj}{hyp}, \quad \tan\theta = \frac{opp}{adj}$$

We can also write the reciprocal functions as

$$\csc\theta = \frac{hyp}{opp}, \quad \sec\theta = \frac{hyp}{adj}, \quad \text{and } \cot\theta = \frac{adj}{opp}$$

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(3)

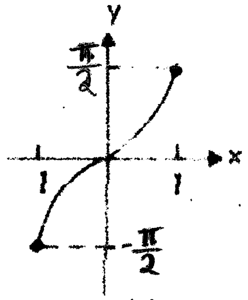
$y = \sin^{-1}(x)$	$y = \cos^{-1}(x)$	$y = \tan^{-1}(x)$
dom $= [-1, 1]$	dom $= [-1, 1]$	dom $= (-\infty, \infty)$
rng $= [-\frac{\pi}{2}, \frac{\pi}{2}]$	rng $= [0, \pi]$	rng $= (-\frac{\pi}{2}, \frac{\pi}{2})$

Inverse trig functions are angles.

Inverse Sine returns an angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (ie, Q1 or Q4)

Inverse Cosine returns an angle in $[0, \pi]$ (ie, Q1 or Q2)

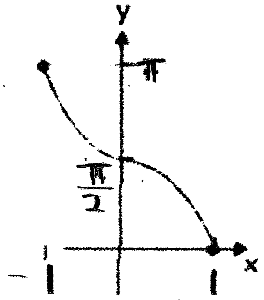
Inverse tangent returns an angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$ (ie, Q1 or Q4)



$f(x) = \sin^{-1}(x)$

Domain : $-1 \leq x \leq 1$

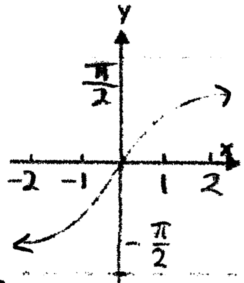
Range : $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



$f(x) = \cos^{-1}(x)$

Domain : $-1 \leq x \leq 1$

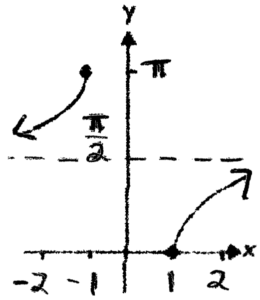
Range : $0 \leq y \leq \pi$



$f(x) = \tan^{-1}(x)$

Domain $-\infty < x < \infty$

Range $-\frac{\pi}{2} < y < \frac{\pi}{2}$

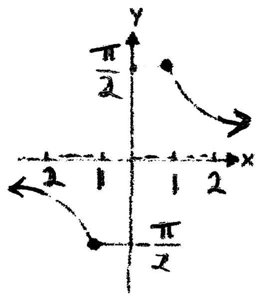


$f(x) = \sec^{-1}(x)$

Domain : $x \leq -1$ or $x \geq 1$

Range : $0 \leq y \leq \pi$ $y \neq \frac{\pi}{2}$

$y = \frac{\pi}{2}$ is a horizontal asymptote

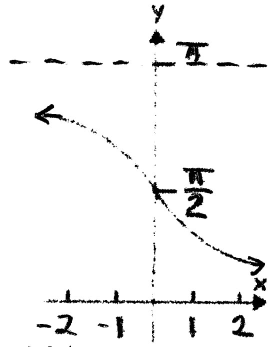


$f(x) = \csc^{-1}(x)$

Domain : $x \leq -1$ or $x \geq 1$

Range : $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ $y \neq 0$

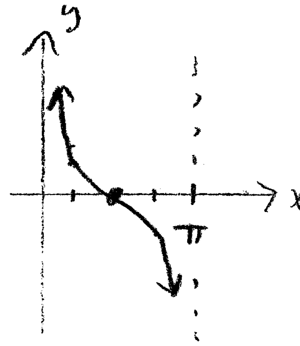
$y=0$ is a horizontal asymptote



$f(x) = \cot^{-1}(x)$

Range : $0 < y < \pi$

Domain : $-\infty < x < \infty$

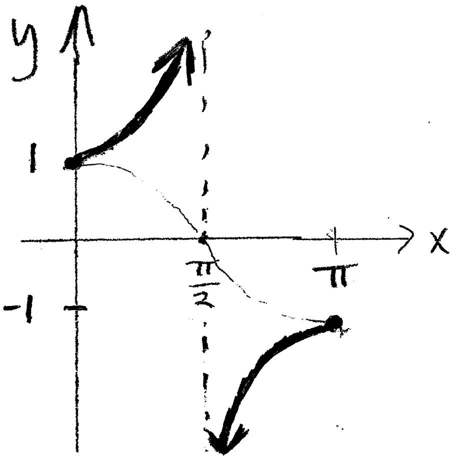


$f(x) = \cot(x)$

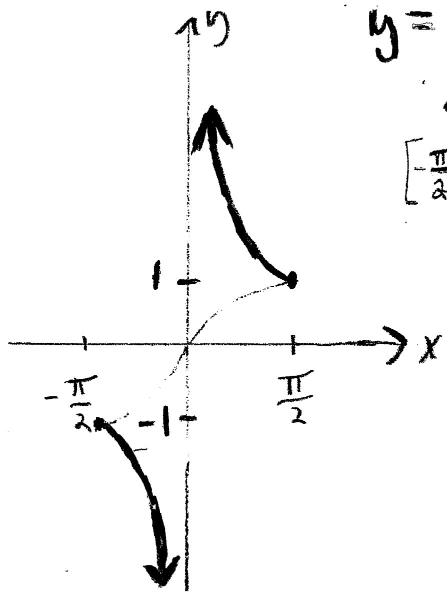
$0 < x < \pi$

$-\infty < y < \infty$

$y = \sec(x)$ on domain $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$



$y = \csc(x)$ on domain $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$



Inverse function identities

① $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$

② $\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$

True for all x in $(-\infty,-1]\cup[1,\infty)$

③ $$\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) + 180^\circ, & \text{if } x<0 \text{ (that is, if } \cot^{-1}(x) \text{ is in Q2)} \\ \text{undefined}, & \text{if } x=0 \\ \tan^{-1}\left(\frac{1}{x}\right), & \text{if } x>0 \text{ (that is, if } \cot^{-1}(x) \text{ is in Q1)} \end{cases}$$

Examples

$\sec^{-1}(4) = \cos^{-1}\left(\frac{1}{4}\right) \approx 75.52^\circ$

$\csc^{-1}(5) = \sin^{-1}(1/5) \approx 11.53^\circ$

$\cot^{-1}(5) = \tan^{-1}\left(\frac{1}{5}\right) \approx 11.31^\circ$

$\cot^{-1}(-2) = \tan^{-1}\left(-\frac{1}{2}\right) + 180^\circ \approx -26.57^\circ + 180^\circ = 153.43^\circ$

(18) Find the exact value of $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ without a calculator.

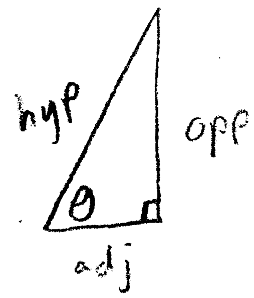
Soln/ Note that for questions of this variety, where the innermost function is an inverse trig function, I will use the "right triangle trig" approach.

We know inverse trig functions represent angles.

$\boxed{\text{Step 1}}$ Let $\theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. Then, $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\theta$,

and $\sin\theta = -\frac{\sqrt{3}}{2}$ (by the property that defines inverse functions ie., since $f(f^{-1}(x)) = x$)

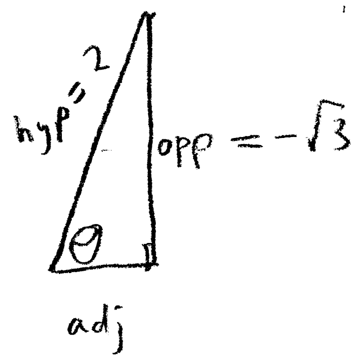
$\boxed{\text{Step 2}}$ Draw a right triangle and locate an acute angle, θ. Label the sides with "opp" "hyp" and "adj".



$\boxed{\text{Step 3}}$ Recall SOH-CAH-TOA.

We know $\sin\theta = \frac{-\sqrt{3}}{2}$ and

we know $\sin\theta = \frac{\text{opp}}{\text{hyp}}$.



Use these facts to label the hypotenuse and the side opposite the angle, θ with exact values.

(18) cont'd

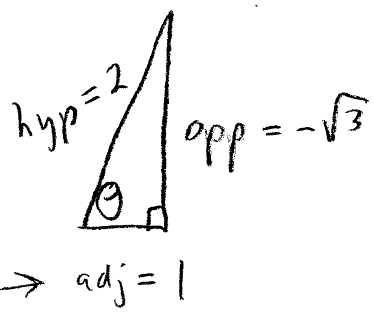
Step 4 Use the pythagorean theorem to find the missing side length

$$2^2 = a^2 + (-\sqrt{3})^2$$

$$4 = a^2 + 3$$

$$a^2 = 1$$

$$a = 1$$ ← (always take the positive square root at this step)



Step 5 Take the cosine of theta now.

$$\begin{pmatrix} \text{given} \\ \text{expression} \end{pmatrix} = \cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \cos\theta \qquad \left(\text{since we agreed that } \theta = \sin^{-1}(-\sqrt{3}/2)\right)$$

$$= \frac{\text{adj}}{\text{hyp}} \qquad (\text{SOH - CAH - TOA})$$

$$= \boxed{\frac{1}{2}}$$

Note: since $\sin\theta = \frac{-\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{-2}$, we could have labeled our triangle with hyp $= -2$, and opp $= \sqrt{3}$. This would give $\cos\theta = -1/2$, but that is a wrong answer!! Guideline: Never label "hyp" as a negative length!!!

19 Find the exact value of $\sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ without a calculator.

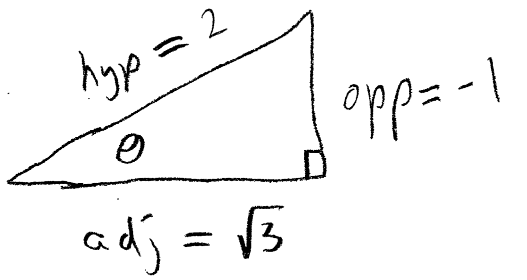
Soln

$\boxed{\text{Step 1}}$ Let $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$.

Then, $\sin\theta = -\frac{1}{2}$ and $\sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \sec(\theta)$.

In other words, we can draw a right triangle with acute angle θ whose sine is $-1/2$. Then, we will take the secant of that angle as our answer.

$\boxed{\text{Steps 2-4}}$ Recall SOH-CAH-TOA. Recall $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{-1}{2}$. (Always put the negative in the numerator)



$$c^2 = a^2 + b^2$$
$$2^2 = a^2 + (-1)^2$$
$$4 = a^2 + 1$$
$$a^2 = 3$$
$$a = \sqrt{3}$$
(always take the positive square root here!!!)

$\boxed{\text{Step 5}}$ The question requires us to find the secant of the angle θ in the right triangle above. Recall that secant and cosine are reciprocals of each other. Recall SOH-CAH-TOA. $\cos\theta = \frac{\text{adj}}{\text{hyp}}$ so $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$
$= \boxed{\frac{2\sqrt{3}}{3}}$

(20) Find the exact value of $\csc\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ without a calculator

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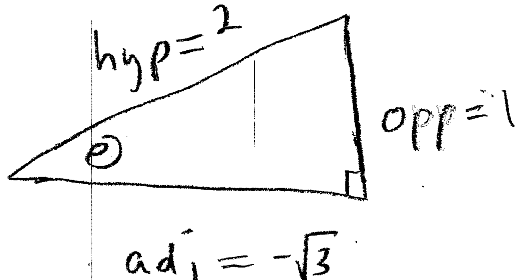
Soln/ Let $\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. Then, $\cos\theta = \frac{-\sqrt{3}}{2}$

and $\csc\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \csc\theta$.



Recall soh-cah-toa, or that $\cos\theta = \frac{adj}{hyp}$ and $\cos\theta = \frac{-\sqrt{3}}{2}$, to help label values on the triangle.

$c^2 = a^2 + b^2$

$2^2 = (\sqrt{3})^2 + b^2$

$4 = 3 + b^2$

$b^2 = 1$

$b = \sqrt{1} = 1$

$\begin{pmatrix}\text{The given}\\ \text{expression}\end{pmatrix} = \csc\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= \csc\theta$

$= \frac{hyp}{opp}$ $\left(\text{We know } \sin\theta = \frac{opp}{hyp}, \text{ so } \csc\theta = \frac{hyp}{opp}\right)$

$= \frac{2}{1} = \boxed{2}$

(21) Find the exact value of $\cos^{-1}\left(\sin\frac{5\pi}{4}\right)$ without a calculator

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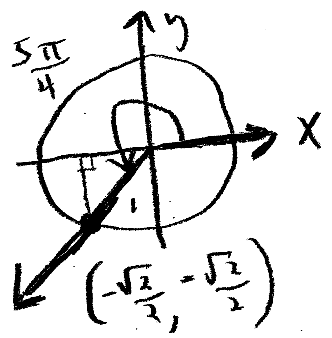
Soln/ Notice that problems 21, 22, 23, 24, 35, 36 have the inverse trig function on the outside and the trig function on the inside. These problems are of a different flavor than problems 9 through 20. Those problems had the inverse trig function as input for a trig function. Evaluate the inner function at the given angle, $\frac{5\pi}{4}$. Then evaluate the outer function. Since the outer function is an inverse trig function and we know that inverse trig functions are angles, the final answer will be an angle. Since we are dealing with inverse cosine, we know that angle will be an angle in the interval $[0, \pi]$.

We know $\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$. Then $\cos^{-1}\left(\sin\frac{5\pi}{4}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.



(21) Find the exact value of $\cos^{-1}\left(\sin \frac{5\pi}{4}\right)$ without a calculator

Soln continued

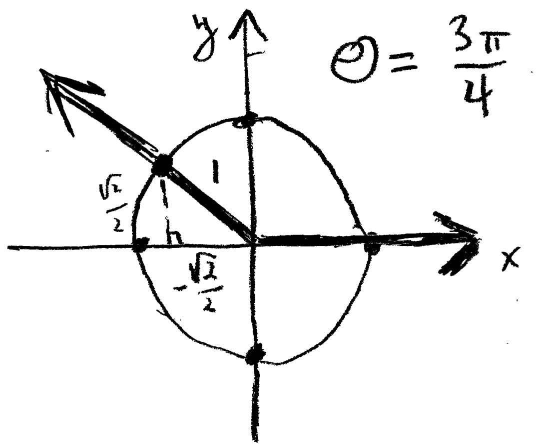
We established that

$$\cos^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

and that $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ was an angle in $[0, \pi]$.

Let θ be that angle. I.e., let $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. Then $\cos\theta = -\frac{\sqrt{2}}{2}$.

Now what angle on the unit circle in the interval $[0, \pi]$ (a Q1 or Q2 angle) has a cosine equal to $-\sqrt{2}/2$?



So,

$$\cos^{-1}\left(\sin \frac{5\pi}{4}\right)$$

$$= \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

$$= \boxed{\frac{3\pi}{4} \text{ (or } 135^\circ)}$$

(22) Find the exact value of $\tan^{-1}\left(\cot \frac{2\pi}{3}\right)$

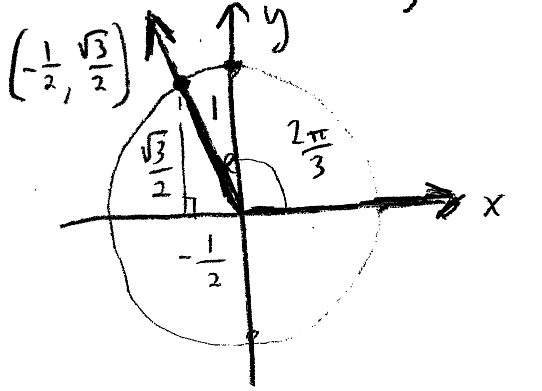
without a calculator

Soln) Step 1 Find $\cot\left(\frac{2\pi}{3}\right)$

Step 2 Find the inverse tangent of this value obtained in step 1.

$\boxed{\text{Step 1}}$ $\cot\left(\frac{2\pi}{3}\right) = \frac{x}{y} = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{2}\cdot\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$

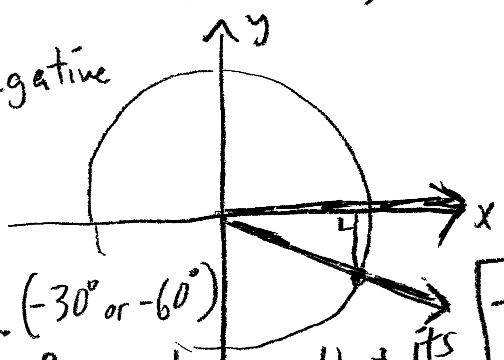
$= -\frac{\sqrt{3}}{3}$



$\boxed{\text{Step 2}}$ $\tan^{-1}\left(\cot\frac{2\pi}{3}\right) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

and we know Inverse tangent is an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$. Then,

$\tan\theta = -\frac{\sqrt{3}}{3}$. Since we have a negative tangent, the angle we are looking for is a negative angle in Q4. Since it has a $\sqrt{3}$ in it, its either $-\frac{\pi}{6}$ or $-\frac{\pi}{3}$. ($-30°$ or $-60°$) We check what the tangent of both angles and see that its $\boxed{-\frac{\pi}{6}}$



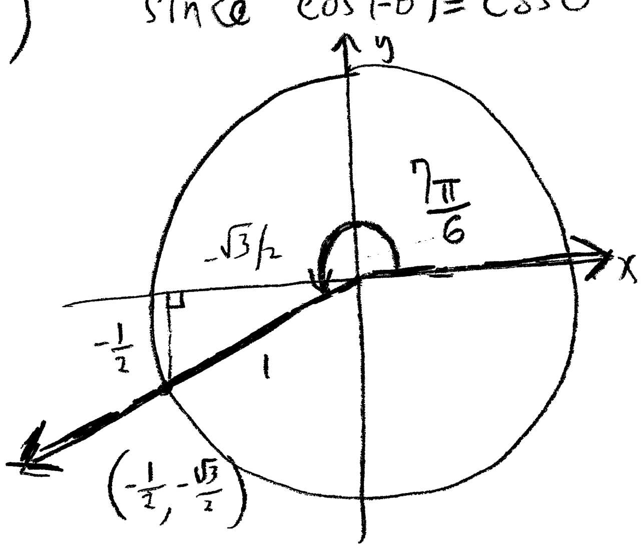
(23) $\sin^{-1}\left(\cos\left(-\frac{7\pi}{6}\right)\right)$

Soln/ ① Find $\cos\left(-\frac{7\pi}{6}\right)$

② Find the inverse sine of this value obtained in step 1

① $\cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right)$ since $\cos(-\theta) = \cos\theta$

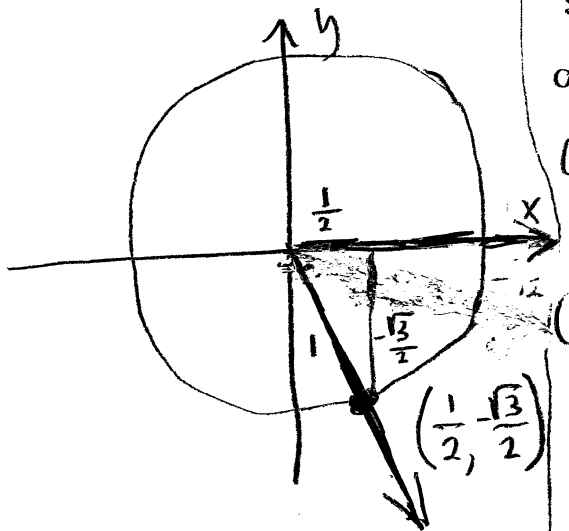
$= \boxed{-\frac{\sqrt{3}}{2}}$



② Let $\theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

then $\sin\theta = -\frac{\sqrt{3}}{2}$ and

θ is a common angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Since we have a negative sine, the angle we are looking for is a negative Q4 angle. Since sine of the angle is $-1/2$, θ is either $-\frac{\pi}{6}$ or $-\frac{\pi}{3}$ ($-30°$ or $-60°$). But, its $\boxed{-\frac{\pi}{3}}$

(30) $\csc\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$ Find the exact value without a calculator

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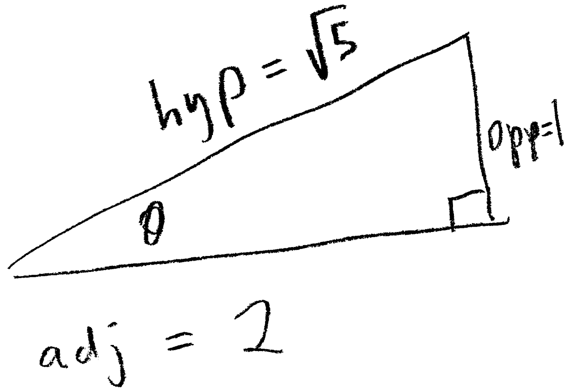
Soln/ Let $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, then $\tan\theta = \frac{1}{2}$



and $\tan\theta = \frac{opp}{adj}$

$c^2 = 2^2 + 1^2$

$c^2 = 4 + 1$

$c^2 = 5$

$c = \sqrt{5}$

Then, $\csc\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$

$= \csc\theta$

$= \frac{hyp}{opp}$

$= \frac{\sqrt{5}}{1} = \boxed{\sqrt{5}}$

and $\sin\theta = \frac{opp}{hyp}$ so

$\csc\theta = \frac{hyp}{opp}$.

(31) $\sin\left(\tan^{-1}(-3)\right)$ Find the exact value without a calculator

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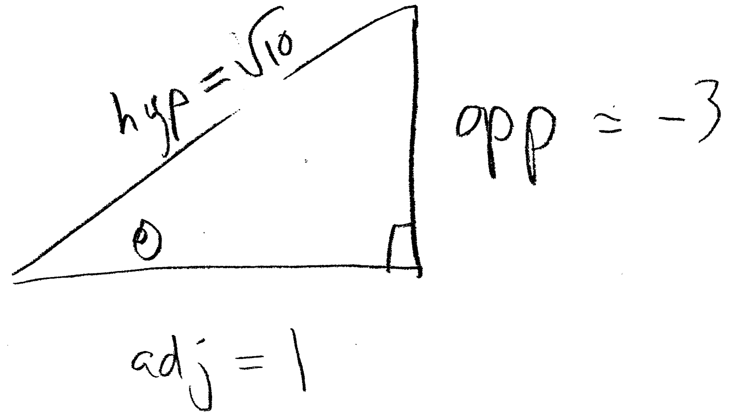
Soln/ Let $\theta = \tan^{-1}(-3)$, then $\tan\theta = -3 = \frac{-3}{1}$

and $\tan\theta = \frac{opp}{adj}$



$$c^2 = a^2 + b^2$$
$$c^2 = 1^2 + (-3)^2$$
$$c^2 = 1 + 9$$
$$c^2 = 10$$
$$c = \sqrt{10}$$

We have to know what quadrant our angle is in to correctly label the right triangle. We know $\tan^{-1}(-3)$ is an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, a Q1 angle or a negative angle in Q4. But $\tan\theta = -3$, a negative value. This tells us our angle θ is in Q4. Moreover, we want to find the sine of θ and we know $\sin\theta$ is negative for angles in Q4. Thus, we expect our answer to be negative. Since $\sin\theta = \frac{opp}{hyp}$, we cannot label the adjacent side length as -1. The negative goes to the opposite side.

Then $\sin\left(\tan^{-1}(-3)\right) = \sin\theta = \frac{opp}{hyp} = \frac{-3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \boxed{\frac{-3\sqrt{10}}{10}}$

<u>Note</u>: had we labeled the opp side as 3 and not -3, we would have obtained the wrong answer.

Ex Find the exact value of $\cot^{-1}(1)$

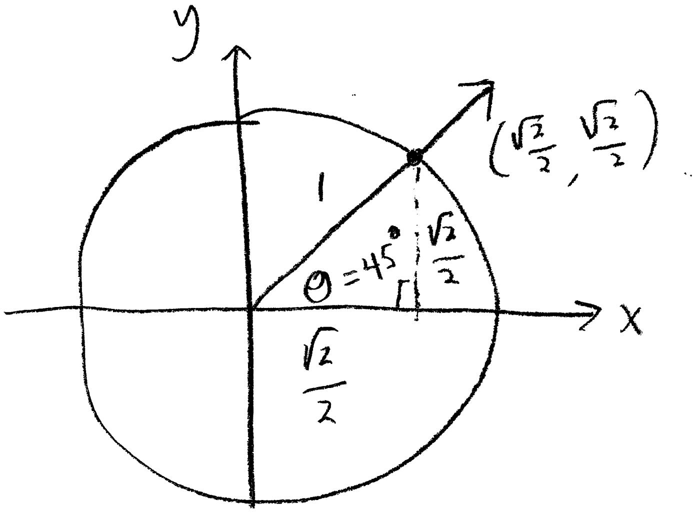
without a calculator

soln/ Let $\theta = \cot^{-1}(1)$ then $\cot\theta = 1$

and $\tan\theta = 1$ by the reciprocal property.

Then $\theta = \tan^{-1}(1)$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
(a Q1 or Q4 angle)

Since tangent is positive, θ must be a common angle in Q1 in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



So,

$$\theta = \cot^{-1}(1)$$

$$= \tan^{-1}(1)$$

$$= \boxed{45^\circ \text{ or } \frac{\pi}{4}}$$

(37) Find the exact value of $\cot^{-1}\sqrt{3}$ without a calculator

---

Soln) First, note that $\cot^{-1}\sqrt{3} = \cot^{-1}(\sqrt{3})$.

Let $\theta = \cot(-\sqrt{3})$, then $\cot\theta = -\frac{\sqrt{3}}{1}$

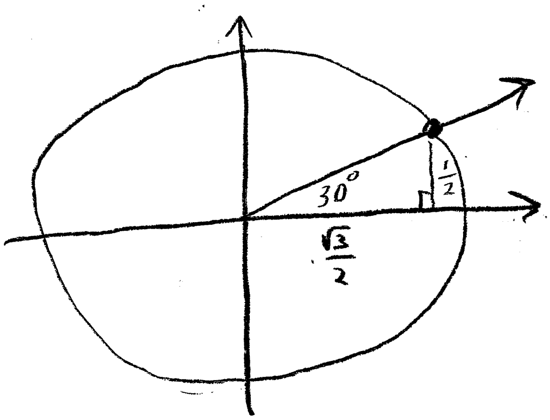
and $\tan\theta = \frac{1}{\sqrt{3}}$, by the reciprocal property.

then $\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

So, we are looking for angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Since tangent is positive, we are looking for a QI angle; again either 30° or 60° since we have a $\sqrt{3}$.

Then, since $\tan(30^\circ) = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$,



$$\theta = \cot^{-1}(\sqrt{3})$$
$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \boxed{30^\circ \text{ or } \frac{\pi}{6}}$$

(39) Find the exact value of $\csc^{-1}(-1)$ without a calculator

---

Soln/ (I will use inverse sine)

Let $\theta = \csc^{-1}(-1)$ then $\csc\theta = -1$

and $\sin\theta = -1$ by the reciprocal property.

Then, $\theta = \sin^{-1}(-1)$ by the properties of inverses.

Also, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

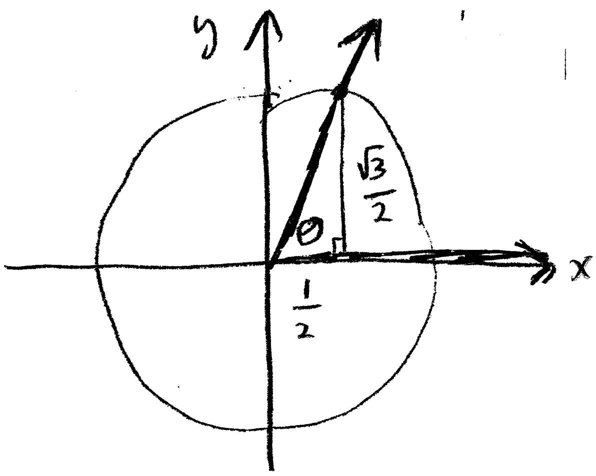
Then,
$$\theta = \csc^{-1}(-1)$$
$$= \sin^{-1}(-1)$$
$$= \boxed{-\frac{\pi}{2}}$$

Example Find the exact value of $\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$ without a calculator

---

Soln/ Let $\theta = \csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$ then $\csc\theta = \frac{2}{\sqrt{3}}$

by properties of inverse functions. If $\csc(\theta) = \frac{2}{\sqrt{3}}$ then $\sin\theta = \frac{\sqrt{3}}{2}$ by the reciprocal property. Then, since $\sin\theta = \frac{\sqrt{3}}{2}$, then $\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and θ is angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Then, $\theta = \csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \boxed{60^\circ \text{ or } \frac{\pi}{3}}$$

Example $\sec^{-1}(-\sqrt{2})$ Find the exact value without a calculator

---

Soln/ (I will use inverse cosine)

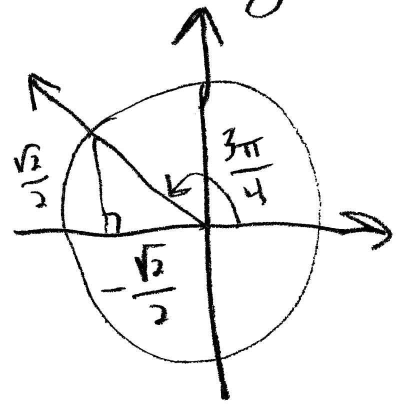
Let $\theta = \sec^{-1}(-\sqrt{2})$ then $\sec(\theta) = -\sqrt{2}$,

and $\cos(\theta) = \dfrac{1}{-\sqrt{2}}$ by the reciprocal property. Then, $\cos\theta = -\dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

and since $\cos\theta = -\dfrac{\sqrt{2}}{2}$, $\theta = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

and θ is an angle in the interval $[0, \pi]$.

What angle in $[0,\pi]$ has cosine equal to $-\dfrac{\sqrt{2}}{2}$?



Then $\theta = \sec^{-1}(-\sqrt{2})$

$= \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

$= \boxed{\dfrac{3\pi}{4} \text{ or } 135^\circ}$

(41) Find the exact value of $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$ without a calculator

---

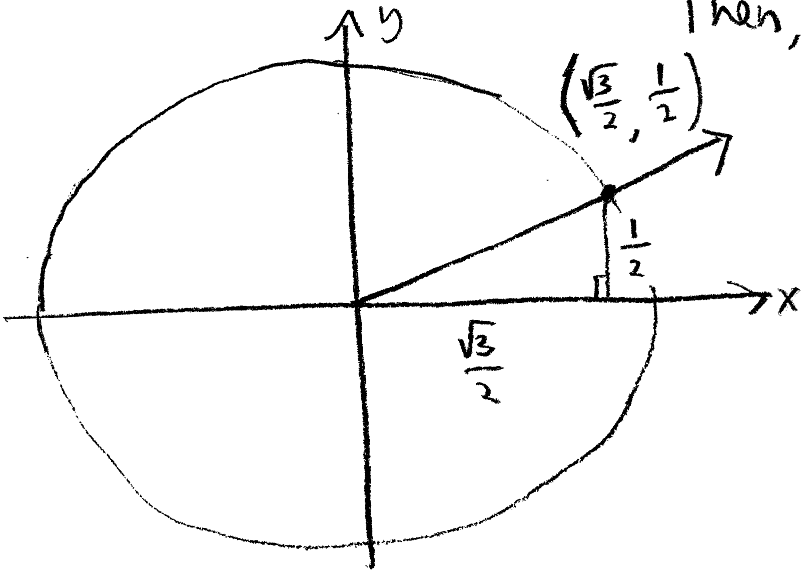
Soln/ Let $\theta = \sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$ then $\sec\theta = \frac{2\sqrt{3}}{3}$.

and $\cos(\theta) = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{2\cdot 3} = \frac{\sqrt{3}}{2}$, or

equivalently $\cos\theta = \frac{\sqrt{3}}{2}$. Then, $\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

and θ is an angle in the interval $[0, \pi]$.

Since Cosine is positive here, the angle we are looking for is A common angle in Q1.



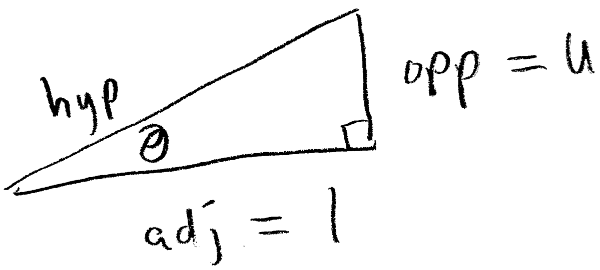
Then, $\theta = \sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$

$= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{30^\circ \text{ or } \frac{\pi}{6}}$

Ex Write $\sin(\tan^{-1} u)$ as an algebraic expression containing u.

Let $\theta = \tan^{-1}(u)$ then $\tan\theta = u = \frac{u}{1} = \frac{opp}{adj}$



$$c^2 = a^2 + b^2$$

$$c^2 = 1 + u^2$$

$$c = \sqrt{1+u^2}$$

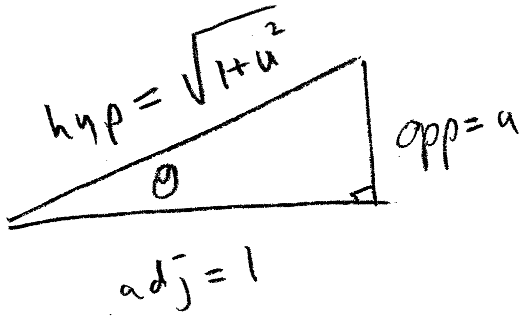
Then,

$$\sin(\tan^{-1} u)$$

$$= \sin(\theta)$$

$$= \frac{opp}{hyp}$$

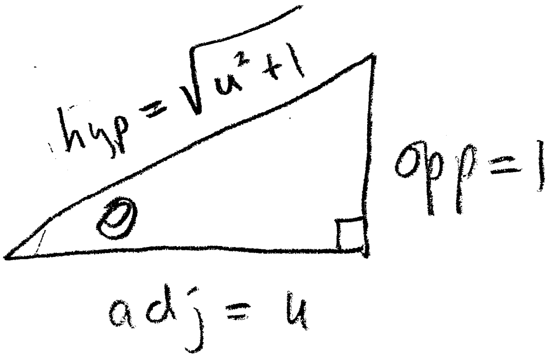
$$= \boxed{\frac{u}{\sqrt{1+u^2}}}$$



(Ex) Write $\sin(\cot^{-1} u)$ as an algebraic expression in u.

Soln/ First note that $\cot^{-1} u = \cot^{-1}(u)$.

Let $\theta = \cot^{-1}(u)$. Then, $\cot\theta = u = \frac{u}{1} = \frac{adj}{opp}$.



$$c^2 = A^2 + b^2$$
$$c^2 = u^2 + 1^2$$
$$c = \sqrt{u^2+1}$$

Then $\sin(\cot^{-1} u)$

$$= \sin\theta$$

$$= \frac{opp}{hyp} = \boxed{\frac{1}{\sqrt{u^2+1}}}$$

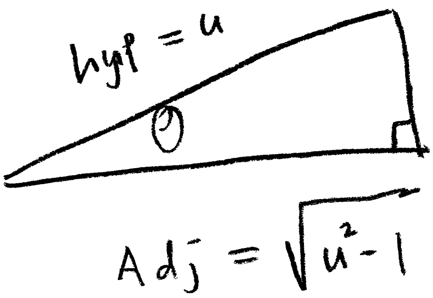
Ex Write $\tan(\csc^{-1} u)$ as an algebraic expression in u

---

soln

Let $\theta = \csc^{-1}(u)$ then $\csc\theta = u$.

or $\csc\theta = \frac{u}{1} = \frac{\text{hyp}}{\text{opp}}$



$c^2 = a^2 + b^2$

$u^2 = a^2 + 1^2$

$a^2 = u^2 - 1$

$a = \sqrt{u^2-1}$

Then, $\tan(\csc^{-1} u)$

$= \tan\theta$

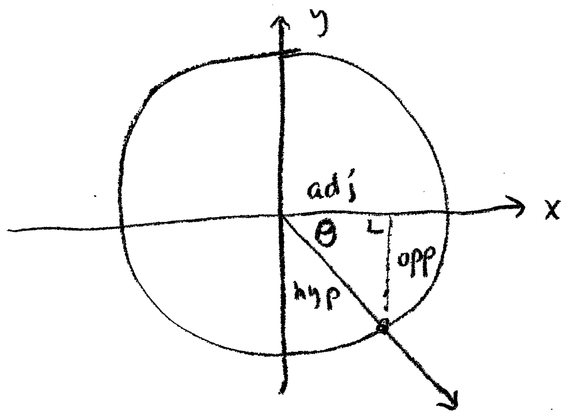
$= \frac{\text{opp}}{\text{adj}} = \boxed{\frac{1}{\sqrt{u^2-1}}}$

$\boxed{\text{Guidelines}}$ for using a right-triangle trig approach to finding exact values of expressions such as $\sin(\cos^{-1}(x))$, $\cos(\sin^{-1}(x))$, $\sec(\tan^{-1}(x))$, etc. where the inner function is an inverse trig function

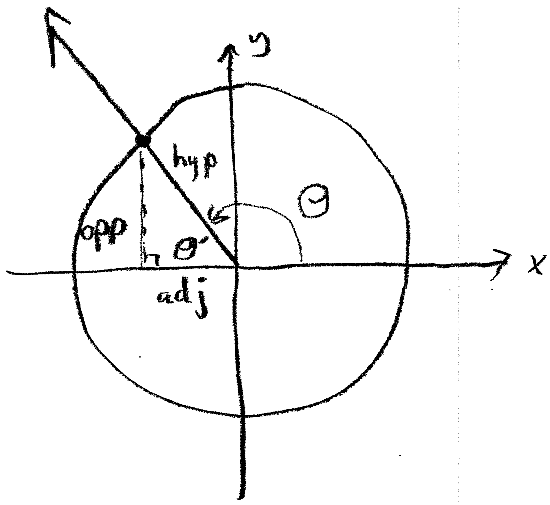
---

① Always label your right triangle's hypotenuse with a positive value.

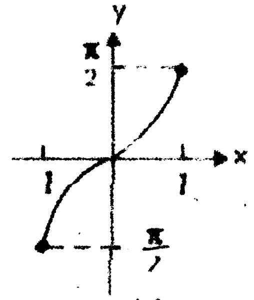
② Special cases exist when the innermost function is $\tan^{-1}(x)$ or $\cot^{-1}(x)$ <u>and $x<0$.</u> ($\underline{\text{Ex}}$ $\cos(\tan^{-1}(-\sqrt{3}))$ or $\sin(\cot^{-1}(-\sqrt{3}))$) When you get a problem like this, you have to make a decision to label either the opp side or the adj side with a negative value. Find out what quadrant your angle is in to determine this. When the innermost function is inverse tangent and the input is negative, then the angle θ terminates in Q4, and the "opp" side should be negative. When the innermost function is inverse cotangent and the input number is negative, the angle θ terminates in Q2, and the "adj" side should be labeled As a negative quantity.



Q4 Angle



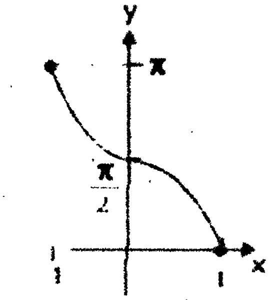
Q2 Angle



$f(x) = \sin^{-1}(x)$

Domain : $-1 \leq x \leq 1$

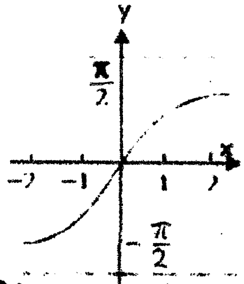
Range : $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



$f(x) = \cos^{-1}(x)$

Domain : $-1 \leq x \leq 1$

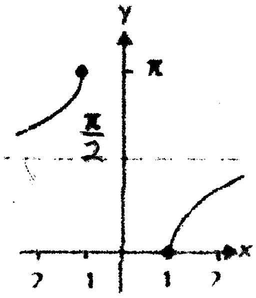
Range : $0 \leq y \leq \pi$



$f(x) = \tan^{-1}(x)$

Domain $-\infty < x < \infty$

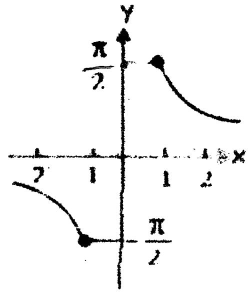
Range $-\frac{\pi}{2} < y < \frac{\pi}{2}$



$f(x) = \sec^{-1}(x)$

Domain : $x \leq -1$ or $x \geq 1$

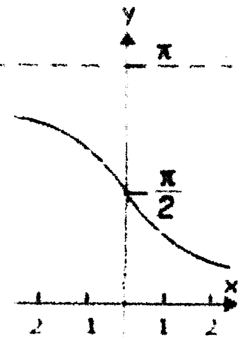
Range : $0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$



$f(x) = \csc^{-1}(x)$

Domain : $x \leq -1$ or $x \geq 1$

Range : $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



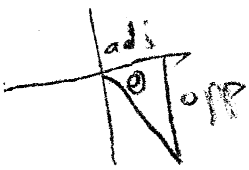
$f(x) = \cot^{-1}(x)$

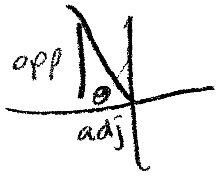
Range : $0 < y < \pi$

Domain : $-\infty < x < \infty$

$$\left.\begin{aligned} \sec^{-1}(x) &= \cos^{-1}\left(\tfrac{1}{x}\right) \\ \csc^{-1}(x) &= \sin^{-1}\left(\tfrac{1}{x}\right) \end{aligned}\right\} \text{for all } x \text{ in } (-\infty,-1] \cup [1,\infty)$$

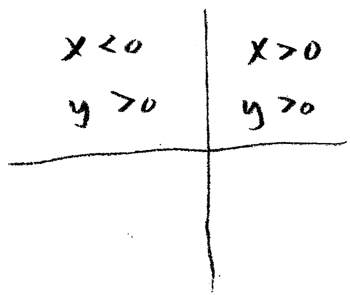
but





$$\cot^{-1}(x) \neq \tan^{-1}\left(\tfrac{1}{x}\right)$$

this is only true when the angle is in Q1. We need to remember that the range of $\cot^{-1}(x)$ is $(0,\pi)$ the same as $\cos^{-1}(x)$.



$\cot\theta = \frac{x}{y}$

$$\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\tfrac{1}{x}\right), \text{ if } \theta \in Q1 \text{ (if } x>0) \\ \tan^{-1}\left(\tfrac{1}{x}\right) + 180^\circ \text{ if } \theta \in Q2 \text{ (if } x<0) \end{cases}$$

$\cot\theta < 0$ in Q2

$\cot\theta > 0$ in Q1