Inverse function identities

① $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$

② $\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$

True for all x in $(-\infty,-1]\cup[1,\infty)$

③ $$\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) + 180^\circ, & \text{if } x<0 \text{ (that is, if } \cot^{-1}(x) \text{ is in Q2)} \\ \text{undefined}, & \text{if } x=0 \\ \tan^{-1}\left(\frac{1}{x}\right), & \text{if } x>0 \text{ (that is, if } \cot^{-1}(x) \text{ is in Q1)} \end{cases}$$

Examples

$\sec^{-1}(4) = \cos^{-1}\left(\frac{1}{4}\right) \approx 75.52^\circ$

$\csc^{-1}(5) = \sin^{-1}(1/5) \approx 11.53^\circ$

$\cot^{-1}(5) = \tan^{-1}\left(\frac{1}{5}\right) \approx 11.31^\circ$

$\cot^{-1}(-2) = \tan^{-1}\left(-\frac{1}{2}\right) + 180^\circ \approx -26.57^\circ + 180^\circ = 153.43^\circ$

$\boxed{\text{Guidelines}}$ for using a right-triangle trig approach to finding exact values of expressions such as $\sin(\cos^{-1}(x))$, $\cos(\sin^{-1}(x))$, $\sec(\tan^{-1}(x))$, etc. where the inner function is an inverse trig function

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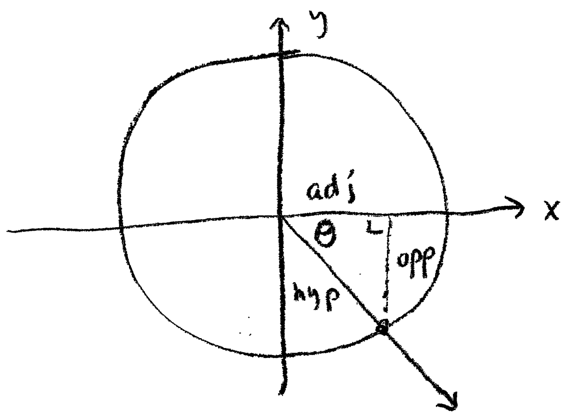
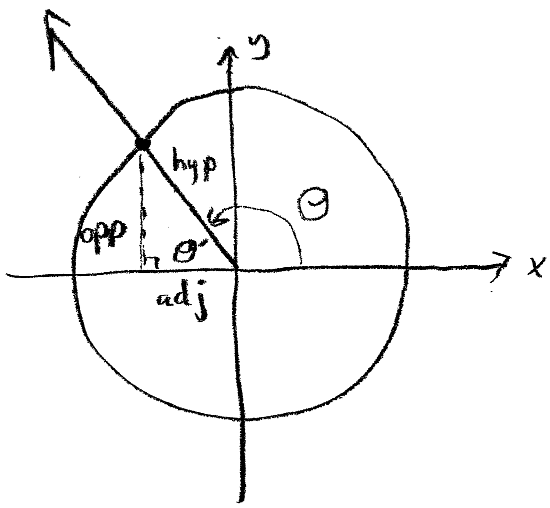
① Always label your right triangle's hypotenuse with a positive value.

② Special cases exist when the innermost function is $\tan^{-1}(x)$ or $\cot^{-1}(x)$ <u>and $x<0$</u>. (<u>Ex</u> $\cos(\tan^{-1}(-\sqrt{3}))$ or $\sin(\cot^{-1}(-\sqrt{3}))$) When you get a problem like this, you have to make a decision to label either the opp side or the adj side with a negative value. Find out what quadrant your angle is in to determine this. When the innermost function is inverse tangent and the input is negative, then the angle θ terminates in Q4, and the "opp" side should be negative. When the innermost function is inverse cotangent and the input number is negative, the angle θ terminates in Q2, and the "adj" side should be labeled as a negative quantity.



Q4 Angle



Q2 Angle

(18) Find the exact value of $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ without a calculator.

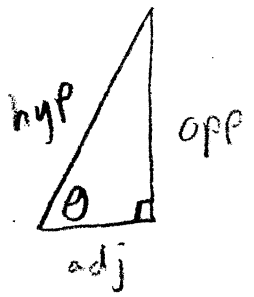
Soln/ Note that for questions of this variety, where the innermost function is an inverse trig function, I will use the "right triangle trig" approach.

We know inverse trig functions represent angles.

$\boxed{\text{Step 1}}$ Let $\theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. Then, $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\theta$,

and $\sin\theta = -\frac{\sqrt{3}}{2}$ (by the property that defines inverse functions ie., since $f(f^{-1}(x)) = x$)

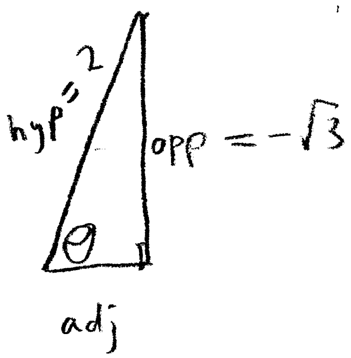
$\boxed{\text{Step 2}}$ Draw a right triangle and locate an acute angle, θ. Label the sides with "opp" "hyp" and "adj".



$\boxed{\text{Step 3}}$ Recall SOH-CAH-TOA.

We know $\sin\theta = \frac{-\sqrt{3}}{2}$ and

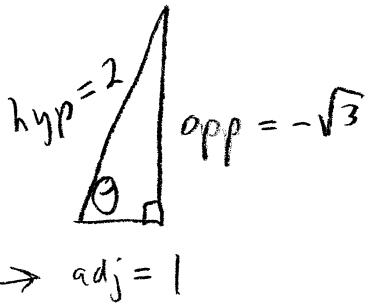
we know $\sin\theta = \frac{\text{opp}}{\text{hyp}}$.



Use these facts to label the hypotenuse and the side opposite the angle, θ with exact values.

(18) cont'd

Step 4 Use the pythagorean theorem to find the missing side length



$2^2 = a^2 + (-\sqrt{3})^2$

$4 = a^2 + 3$

$a^2 = 1$

$a = 1$ « (always take the positive square root at this step)

Step 5 Take the cosine of theta now.

$$\left(\begin{array}{c}\text{given}\\ \text{expression}\end{array}\right) = \cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \cos\theta \qquad \left(\text{since we agreed that } \theta = \sin^{-1}(-\sqrt{3}/2)\right)$$

$$= \frac{\text{adj}}{\text{hyp}} \qquad (\text{SOH-CAH-TOA})$$

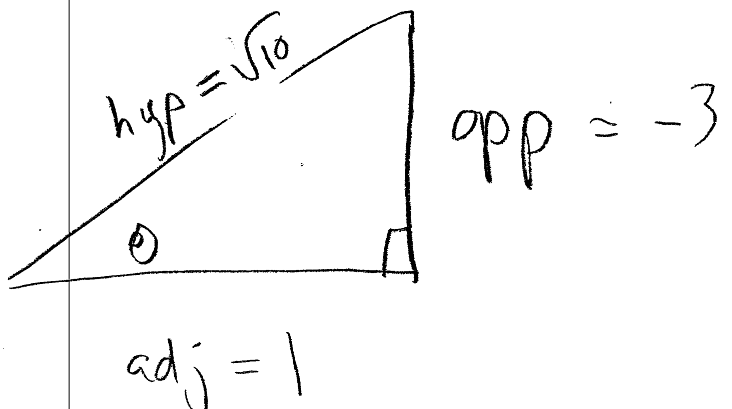
$$= \boxed{\frac{1}{2}}$$

Note: since $\sin\theta = \frac{-\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{-2}$, we could have labeled our triangle with hyp $= -2$, and opp $= \sqrt{3}$. This would give $\cos\theta = -1/2$, but that is a wrong answer!! Guideline: Never label "hyp" as a negative length!!!

(31) $\sin\left(\tan^{-1}(-3)\right)$ Find the exact value without a calculator

soln/ Let $\theta = \tan^{-1}(-3)$, then $\tan\theta = -3 = \frac{-3}{1}$

and $\tan\theta = \frac{\text{opp}}{\text{adj}}$,



$$c^2 = a^2 + b^2$$
$$c^2 = 1^2 + (-3)^2$$
$$c^2 = 1 + 9$$
$$c^2 = 10$$
$$c = \sqrt{10}$$

We have to know what quadrant our angle is in to correctly label the right triangle. We know $\tan^{-1}(-3)$ is an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, a Q1 angle or a negative angle in Q4. But $\tan\theta = -3$, a negative value. This tells us our angle θ is in Q4. Moreover, we want to find the sine of θ and we know $\sin\theta$ is negative for angles in Q4. Thus, we expect our answer to be negative. Since $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, we cannot label the adjacent side length as -1. The negative goes to the opposite side.

Then $\sin\left(\tan^{-1}(-3)\right) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{-3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \boxed{\frac{-3\sqrt{10}}{10}}$

__Note__: had we labeled the opp side as 3 and not -3, we would have obtained the wrong answer.

(43) Find the exact value of

$\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

Let $\theta = \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ then $\theta \in (0, \pi)$, the range of the inverse cotangent. Recall that

$$\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) + 180^\circ, & \text{if } x<0 \ \left(\text{if } \cot^{-1}(x) \in Q2\right) \\ \text{undefined if } x & \text{if } x = 0 \\ \tan^{-1}\left(\frac{1}{x}\right), & \text{if } x>0 \ \left(\text{If } \cot^{-1}(x) \in Q1\right) \end{cases}$$

Since $-\frac{\sqrt{3}}{3}$ is less than zero, we know our angle θ is a Q2 angle. To find this angle we first find the angle $\tan^{-1}\left(\frac{1}{-\frac{\sqrt{3}}{3}}\right)$ then add 180° to it.

Let $\theta' = \tan^{-1}\left(\frac{1}{-\frac{\sqrt{3}}{3}}\right) = \tan^{-1}\left(-\frac{3}{\sqrt{3}}\right) = \tan^{-1}\left(-\frac{3}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}\right)$

$= \tan^{-1}\left(-\frac{3\sqrt{3}}{3}\right) = \tan^{-1}\left(-\sqrt{3}\right).$

That is, let $\theta' = \tan^{-1}(-\sqrt{3})$. Then $\theta' \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\tan\theta' = -\sqrt{3}$.

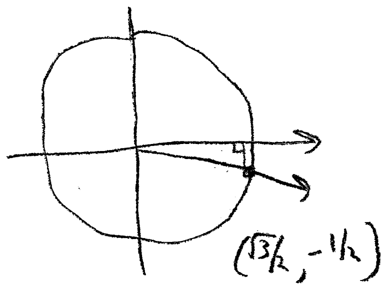
(43) (continued)

Since $\theta' \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, θ' is a Q1 or negative angle in Q4.

But tangent is positive in Q1, and $\tan\theta' = -\sqrt{3} < 0$, so θ' is a negative Q4 angle. Moreover, θ' is either $-30°$ or $-60°$ since we are told $\tan\theta'$ has a "$\sqrt{3}$" in it.

<u>Check both:</u>

$$\tan(-30°) = \frac{y}{x} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$



and $\tan(-60°) = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$

so $\theta' = -60°$.

Then θ, our original angle in Q2 is

$$\theta = \theta' + 180° = -60° + 180° = \boxed{120° \text{ or } \frac{2\pi}{3}}$$