Inverse Trig Functions **Name:** ________________________

Definition 1. *A function $f(x)$ with domain D and range R is a* <u>*one to one function*</u> *if either of the following equivalent conditions are satisfied.*

1. *Whenever $x_1 \neq x_2$ in D then $f(x_1) \neq f(x_2)$ in R.*
2. *Whenever $f(x_1) = f(x_2)$ in R, then $x_1 = x_2$ in D.*

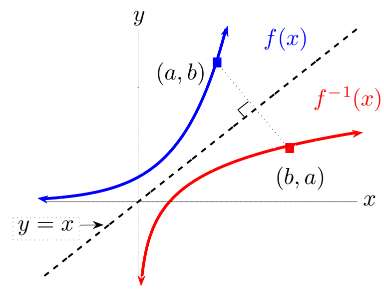
This means that a one to one function has the characteristic: for each functional value $f(x)$ in the range R there corresponds <u>EXACTLY ONE</u> element in the domain D.

Theorem 1. <u>*A function f is one to one*</u> *if and only if every horizontal line intersects the graph of f in at most one point.*

Properties of Inverse Functions

Suppose that f is a one to one function with domain D and range R. Then

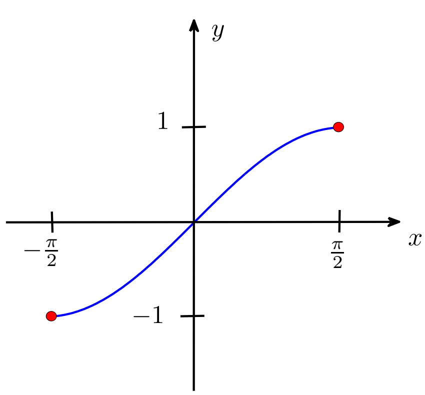
- The inverse function f^{-1} exists, and is unique.
- The domain of f^{-1} is the range of f.
- The range of f^{-1} is the domain of f.
- The statement $f(x) = y$ is equivalent to $f^{-1}(y) = x$
- If x is in the domain of $f(x)$, then $\left(f^{-1} \circ f\right)(x) = x$ and $\left(f \circ f^{-1}\right)(x) = x$



The Inverse Sine Function

The sine function is not one to one since it doesn't pass the horizontal line test. However, if we restrict the domain of the sine function to $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $f(x) = \sin(x)$ becomes one to one so that a unique f^{-1} exists—written $f^{-1}(x) = \sin^{-1}(x)$.

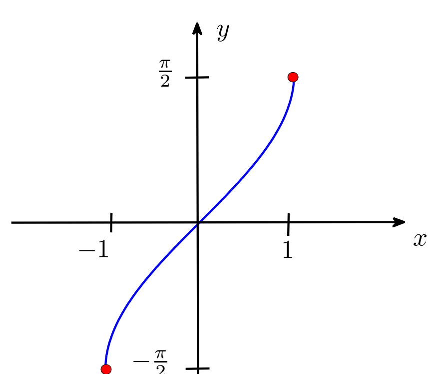
$f(x) = \sin(x)$



$\text{dom}\left(f\right)$ restricted to $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\text{range}\left(f\right) \equiv [-1, 1]$

$f^{-1}(x) = \sin^{-1}(x)$



$\text{dom}\left(f^{-1}\right) \equiv [-1, 1]$

$\text{range}\left(f^{-1}\right) \equiv \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Properties of Inverse Sine

1. For $x \in [-1, 1]$, $\sin\left(\sin^{-1}(x)\right) = x$,

2. For $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}\left(\sin(x)\right) = x$,

3. $\left[\, y = \sin^{-1}(x) \,\right] \Longleftrightarrow \left[\, \sin(y) = x \,\right]$

4. $\sin^{-1}(x) \neq \dfrac{1}{\sin(x)} = \left(\sin(x)\right)^{-1}$

5. The inverse sine of a value in $[-1, 1]$ will return a Q1 or Q4 (negative) angle or be $-\frac{\pi}{2}$ or $\frac{\pi}{2}$.

1. **Find each value.**

(a) $\sin^{-1}\left(\frac{1}{2}\right)$

(b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

(c) $\sin^{-1}\left(\frac{3}{2}\right)$

(d) $\sin^{-1}(-1)$

Use a calculator to find approximate to 3 decimal places.

2. $\sin^{-1}(-0.3)$ 2. ____________

3. $\sin^{-1}(-19)$ 3. ____________

Evaluate without a calculator.

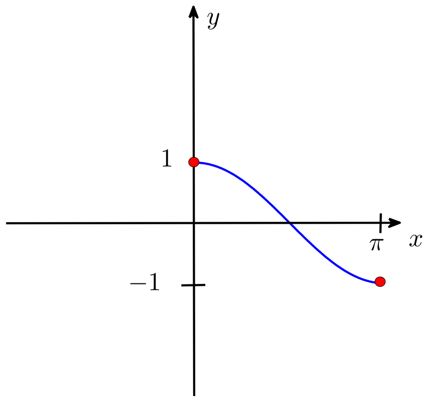
4. $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ 4. ____________

5. $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$ 5. ____________

The Inverse Cosine Function

The cosine function is not one to one since it doesn't pass the horizontal line test. However, if we restrict the domain of the cosine function to $x \in [0, \pi]$, then $f(x) = \cos(x)$ becomes one to one so that a unique f^{-1} exists—written $f^{-1}(x) = \cos^{-1}(x)$.

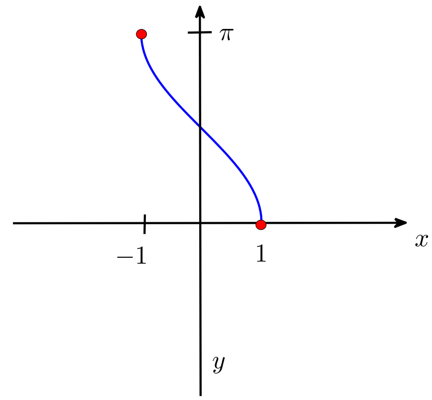
$f(x) = \cos(x)$



$\text{dom}\big(f\big)$ restricted to $x \in [0, \pi]$

$\text{range}\big(f\big) \equiv [-1, 1]$

$f^{-1}(x) = \cos^{-1}(x)$



$\text{dom}\big(f^{-1}\big) \equiv [-1, 1]$

$\text{range}\big(f^{-1}\big) \equiv [0, \pi]$

6. **Find each value.**

(a) $\cos^{-1}\left(\frac{1}{2}\right)$

(b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(c) $\cos^{-1}\left(\frac{3}{2}\right)$

(d) $\cos^{-1}(-1)$

Properties of Inverse Cosine

1. For $x \in [-1, 1]$, $\cos\big(\cos^{-1}(x)\big) = x$,

2. For $x \in [0, \pi]$, $\cos^{-1}\big(\cos(x)\big) = x$,

3. $\Big[\, y = \cos^{-1}(x) \,\Big] \Longleftrightarrow \Big[\, \cos(y) = x \,\Big]$

4. $\cos^{-1}(x) \neq \dfrac{1}{\cos(x)} = \big(\cos(x)\big)^{-1}$

5. The inverse cosine of a value in $[-1, 1]$ will return a Q1 or Q2 angle or be 0 or π.

Use a calculator to find approximate to 3 decimal places.

7. $\cos^{-1}(.8)$ — 7. ____________

8. $\cos^{-1}(-27)$ — 8. ____________

Evaluate without a calculator.

9. $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ — 9. ____________

10. $\cos^{-1}\left(\cos\frac{\pi}{6}\right)$ — 10. ____________

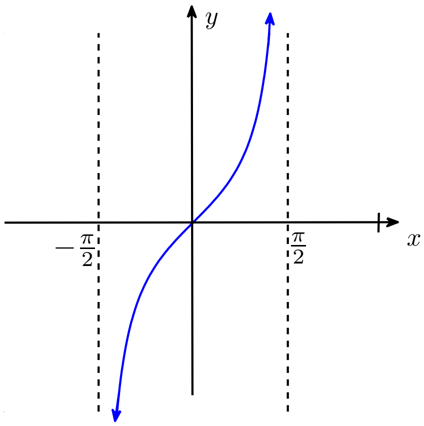
11. $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ — 11. ____________

12. $\sin\left(\cos^{-1}\left(\frac{7}{25}\right)\right)$ — 12. ____________

The Inverse Tangent Function

The tangent function is not one to one since it doesn't pass the horizontal line test. However, if we restrict the domain of the tangent function to $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $f(x) = \tan(x)$ becomes one to one so that a unique f^{-1} exists—written $f^{-1}(x) = \tan^{-1}(x)$.

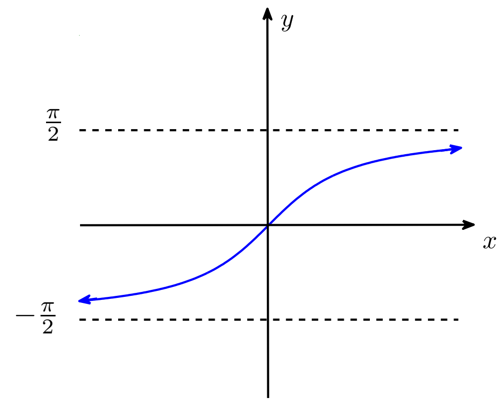
$f(x) = \tan(x)$



$\text{dom}(f)$ restricted to $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\text{range}(f) \equiv (-\infty, \infty)$

$f^{-1}(x) = \tan^{-1}(x)$



$\text{dom}(f^{-1}) \equiv (-\infty, \infty)$

$\text{range}(f^{-1}) \equiv \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

6. **Find each value.**

(a) $\tan^{-1}(0)$

(b) $\tan^{-1}\left(-\sqrt{3}\right)$

(c) $\tan^{-1}(-1)$

Properties of Inverse Tangent

1. For $x \in (-\infty, \infty)$, $\quad \tan\left(\tan^{-1}(x)\right) = x$,
2. For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\quad \tan^{-1}\left(\tan(x)\right) = x$,
3. $\left[\, y = \tan^{-1}(x) \,\right] \Longleftrightarrow \left[\, \tan(y) = x \,\right]$
4. $\tan^{-1}(x) \neq \dfrac{1}{\tan(x)} = \left(\tan(x)\right)^{-1}$
5. The inverse tangent of a value in $(-\infty, \infty)$ will return a Q1 or Q4 (negative) angle.

Find the exact value of the expression (without a calculator), if it is defined.

13. $\tan\left(\cos^{-1}\left(\frac{12}{13}\right)\right)$

14. $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

15. $\sin\left(\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)\right)$