Objectives

- Find square roots without a calculator
- Simplify expressions of the form $\sqrt[n]{a^n}$
- Evaluate radical functions and find the domain of radical functions

Definition 1. *The* ***square of a number*** *is the number times itself.*

For instance, the square of 4 is 16 because 4^2 or $(4)\cdot(4) = 16$. The square of -4 is also 16 because $(-4)^2 = (-4)\cdot(-4) = 16$.

Definition 2. *The reverse process of squaring is* ***finding a square root****.*

For example, a square root of 16 is 4 because $4^2 = 16$. A square root of 16 is also -4 because $(-4)^2 = (-4)\cdot(-4) = 16$.

Theorem 1. *Every positive number has two square roots.*

For instance, the square roots of 25 are 5 and -5.

Definition 3. *We use the symbol* $\sqrt{}$*, called a* ***radical sign****, to indicate the positive (or "principal") square root.*

For example,

$$\sqrt{25} = 5 \text{ because } 5^2 = 25 \text{ and 5 is positive.}$$

$$\sqrt{9} = 3 \text{ because } 3^2 = 9 \text{ and 3 is positive.}$$

Note: it is a common mistake to assume that an expression like $\sqrt{25}$ indicates both square roots, 5 and -5. The expression $\sqrt{25}$ indicates only the positive square root of 25, which is 5. If we want the negative square root, we must use a negative sign in front of the radical sign.

We write the negative square root of 25 as $-\sqrt{25}$ which is -5.

Definition 4. *The* ***square root,*** $\sqrt{}$*, of a positive number a is the positive number b whose square is a. In symbols,*

$$\sqrt{a} = b \quad \textit{if and only if} \quad b^2 = a$$

Guideline: When finding the square root of a positive number, ask yourself: What number times itself is the number under the radical symbol equal to?

Exercises

1. Find the square roots of 169

2. Simplify
   a) $\sqrt{100}$ b) $\sqrt{64}$ c) $-\sqrt{81}$ d) $-\sqrt{121}$ e) $\sqrt{-16}$

Theorem 2. *A square root of a negative number is not a real number.*

Proof. Suppose, on the contrary, that there is a real number that is the square root of a negative number. Let a represent any positive number. Then $-a$ represents any negative number. Consider the square root of $-a$. Suppose it's equal to some real number b. That is, suppose

$$\sqrt{-a} = b$$

Then, by the definition of square root, it must be the case that $b^2 = -a$. However, a positive number times itself is a positive number, and a negative number times itself is a positive number, so $b^2 \neq -a$. Since we derived a contradiction, our assumption that there exists a real number b that is the square root of a negative number must be false. So, there is no real number b whose square is negative a. ■

Definition 5. *The quantity under the radical symbol is called the* ***radicand****. The radical symbol and the radicand make up what is called a* ***radical expression****.*

Definition 6. *A* ***radical function*** *in x is a function defined by an expression containing a root of x.*
The domain of a square root function is the set of real numbers for which the radicand is nonnegative.

For example, let $f(x) = \sqrt{6-2x}$. Then,

$$f(-15) = \sqrt{6-2(-15)} = \sqrt{6+30} = \sqrt{36} = 6$$

and

$$f(11) = \sqrt{6-2(11)} = \sqrt{6-22} = \sqrt{-16}, \text{ not a real number.}$$

The domain of f: Set the radicand greater than or equal to zero. The domain is the solution set (interval) of the inequality.

$$\begin{aligned} 6-2x &\geq 0 \\ 6 &\geq 2x && \text{(add } 2x \text{ to both sides)} \\ 2x &\leq 6 && \text{(symmetric property of inequality)} \\ x &\leq 3 && \text{(divide)} \end{aligned}$$

$\Longrightarrow \quad dom(f) = (-\infty, 3]$

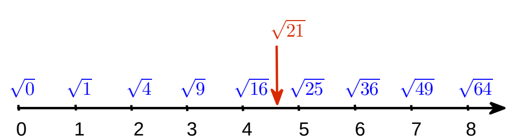
Exercises:

3. Evaluate each square root function for $x = -1$ a) $f(x) = \sqrt{\frac{198-2x}{2}}$ b) $g(x) = \sqrt{\frac{47-x}{3}}$

4. Find the domain of each function given in the previous question.

Definition 7. *Numbers like* $\frac{1}{4}, \frac{4}{25}, 9$ *and 36 are called* ***perfect squares*** *because their square root is a whole number or a fraction.*

A square root such as $\sqrt{21}$ cannot be written as a whole number or a fraction since 21 is not a perfect square. It can be approximated by estimating, by using a table, or by using a calculator. We can however, estimate what two whole numbers $\sqrt{21}$ is between.



Theorem 3. *For any real number a,*

$$\sqrt{a^2} = |a|$$

Exercises:

5. Simplify. Assume that the letters can represent any real number so use absolute value notation when necessary. If a root cannot be simplified, state this.

a) $\sqrt{4x^2}$ b) $\sqrt{(t+4)^2}$ c) $\sqrt{x^2-4x+4}$ d) $\sqrt{a^{10}}$ e) $\sqrt{x^{14}}$ f) $\sqrt{y^{20}}$

Definition 8. *A number is called a **perfect cube number** if it is equal to some rational number raised to the third power. For example, 343 and $\frac{1}{64}$ are perfect cube numbers since*

$$343 = 7^3 \quad \textit{and} \quad \frac{1}{64} = \left(\frac{1}{4}\right)^3$$

Definition 9. *The **cube root**, $\sqrt[3]{}$ of a number a is the number b whose cube is a. In symbols,*

$$\sqrt[3]{a} = b \quad \textit{if} \quad b^3 = a$$

For example,

$$\sqrt[3]{125} = 5 \quad \text{since} \quad 5^3 = 125 \quad \text{and}$$

Guideline: When finding the cube root of a number, ask yourself: What number cubed (raised to the third power) is the number under the radical symbol equal to?

Guideline: The number 3 in the radical expression $\sqrt[3]{125}$ is called the index of the radical. The index of a radical must be a positive integer greater than 1. If no index is written, it is assumed to be 2.

Exercises:

6. Find the given cube root.

a) $\sqrt[3]{8}$ b) $\sqrt[3]{-8}$ c) $-\sqrt[3]{\frac{1}{8}}$ d) $\sqrt[3]{-27}$ e) $\sqrt[3]{x^6}$

f) $\sqrt[3]{y^{21}}$ g) $\sqrt[3]{-t^{15}}$ h) $\sqrt[3]{-8x^6}$ i) $\sqrt[3]{-64x^{21}}$

Definition 10. *$f(x) = \sqrt[3]{x}$ is called the **cube root function**. Since we can find a cube root for any number (positive, negative or zero), $dom(f) = (-\infty, \infty)$.*

Exercises:

7. Graph $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$

8. Find $f(-1)$ and $f(0)$ for $f(x) = \sqrt[3]{6-2x}$

9. Find the domain of $f(x) = \sqrt[3]{6-2x}$

Not every cube root is a perfect cube number. For instance, $\sqrt[3]{7}$ is not a perfect cube number. But $\sqrt[3]{7}$ is a real number. It just so happens that $\sqrt[3]{7}$ is an instance of an irrational number — a number whose decimal equivalent has an infinite number of digits right of the decimal point and those digits occur with no predictable or repeatable pattern. We can, however, find a rational number that best approximates $\sqrt[3]{7}$, which is what computers and calculators do. In this case, $\sqrt[3]{7} \approx 1.913$ since $1.913^3 \approx 7.000755497$.

Definition 11. *The **fourth root**, $\sqrt[4]{}$ of a positive number a is the number b such that*

$$\sqrt[4]{a} = b \quad if \quad b^4 = a$$

For example,

$$\sqrt[4]{16} = 2 \quad \text{since} \quad 2^4 = 16$$

Moreover, $\sqrt[4]{-16}$ is not a real number since there is no real number we can raise to the fourth power and obtain -16.

Exercises:

10. Simplify

a) $\sqrt[4]{1}$ b) $-\sqrt[4]{\dfrac{1}{16}}$ c) $\sqrt[4]{81x^4}$ d) $\sqrt[4]{-16}$ e) $\sqrt[4]{y^8}$ f) $\sqrt[4]{81x^{32}}$

There are also fifth roots, sixth roots, seventh roots, and so on. As a generalization, we call $\sqrt[n]{a}$ the n^{th} root of a. The number n is the index.

Definition 12. *The **n^{th} root**, $\sqrt[n]{a}$, of a positive number a is the number b such that*

$$\sqrt[n]{a} = b \quad if \quad b^n = a$$

We can use the following chart to help summarize the fine print of the definition.

n	a	$\sqrt[n]{a}$	$\sqrt[n]{a^n}$
Even	Positive	Positive	a
	Negative	Not a real number	$-a$
Odd	Positive	Positive	a
	Negative	Negative	a

It needs to be clear that we cannot take an even root of a negative number!!!

Every real number has one root when n is odd. Every positive real number has two real roots when n is even. An even root of a negative number is not a real number.

If n is even, then $\sqrt[n]{a^n} = |a|$.

If n is odd, then $\sqrt[n]{a^n} = a$.

Exercises:

11. Simplify. Use absolute value notation when necessary.

a) $\sqrt[3]{x^{12}}$ b) $\sqrt[4]{(-3x)^4}$ c) $\sqrt[6]{(4-x)^6}$

12. Simplify. Assume that no radicands were formed by raising negative quantities to even powers. Thus, absolute value notation is not necessary.

a) $\sqrt{25x^2}$ b) $-\sqrt[3]{(x+2)^3}$ c) $\sqrt[4]{256y^4}$

A radical function is a function that can be described by a radical expression.

Finding the Domain:

- When the index of the expression is odd, the domain is the set of real numbers, $\mathbb{R}$

- When the index of the expression is even, the domain is the set of values that make the radicand non-negative upon substitution.

Exercises:

13. Find the domains for each function.

a) $f(x) = \sqrt{x-1}$ b) $f(x) = \sqrt[4]{1-2x}$ c) $f(x) = \sqrt[5]{2x-1}$

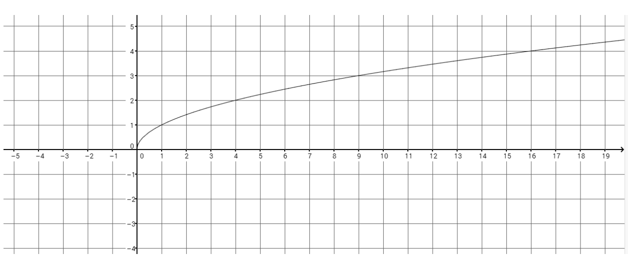
Answers: 1) $-13, 13$ 2a) 10 2b) 8, 2c) -9, 2d) -11 2e) not a real number

3a) 10 3b) 4, 4a) $(-\infty, 99]$ 4b) $(-\infty, 47]$ 5a) $|2x|$ 5b) $|t+4|$,

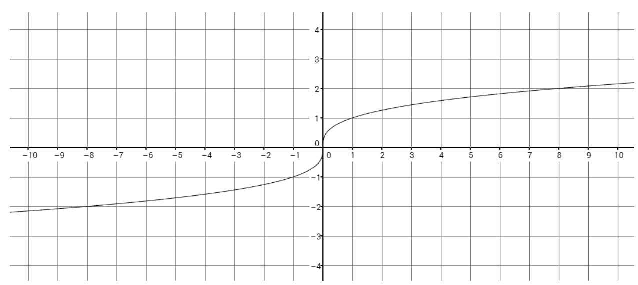
5c) $|x-2|$, 5d) $|a^5|$, 5e) $|x^7|$, 5f) y^{10}, 6a) 2 6b) -2, 6c) $-1/2$,

6d) -3 6e) x^2, 6f) y^7, 6g) $(-t)^5$ 6h) $-2x^2$, 6i) $-4x^7$,

7)



(a) $y = \sqrt{x}$



(b) $y = \sqrt[3]{x}$

8) $2, 0$ 9) $(-\infty, \infty)$

10a) 1 10b) $-1/16$, 10c) $|3x|$, 10d) not a real number 10e) $|3x|$, 10f) $|3x|$,

11a) x^4 11b) $3|x|$, 11c) $|4-x|$,

12a) $5x$ 12b) $-(x+2)$, 12c) $4y$,

13a) $[1, \infty)$ 13b) $(-\infty, 1/2]$, 13c) $(-\infty, \infty)$