Function Combinations & Compositions. One-to-one & Inverse Functions

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Set Intersection

The intersection of two sets A and B, written $A \cap B$, is the set of all elements (numbers) that are in both A and B. The $\cap$ symbol means the word "and."

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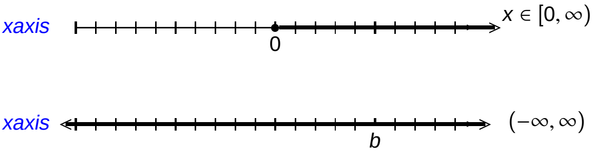

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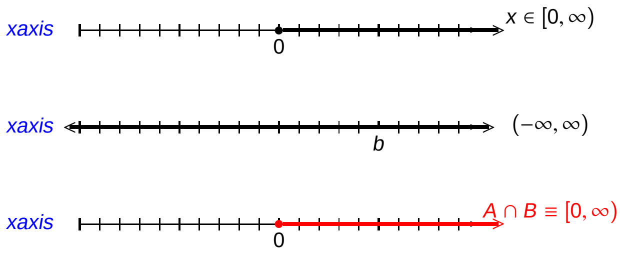

Algebra of Functions

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows:

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) && \text{domain } A\cap B \\ (f-g)(x) &= f(x)-g(x) && \text{domain } A\cap B \\ (f\cdot g)(x) &= f(x)\cdot g(x) && \text{domain } A\cap B \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} && \text{domain } \{x\in A\cap B \mid g(x)\neq 0\} \end{aligned}$$

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Example: Suppose $f(x)=\sqrt{x}$, $g(x)=x^2$ and $h(x)=(f+g)(x)=\sqrt{x}+x^2$.

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$$(f+g)(x) = f(x) + g(x) \qquad \text{domain } A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{domain } A \cap B$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \qquad \text{domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

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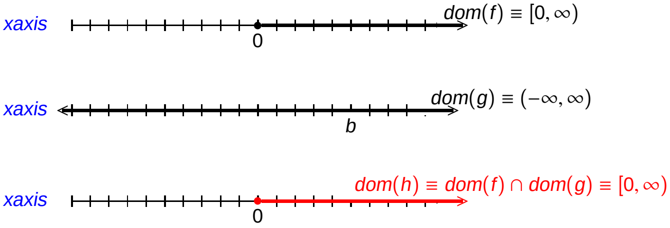

Composition of Functions

If f and g are two functions, the composition of f and g, written $f \circ g$ is defined by the equation

$$f \circ g = f(g(x)),$$

provided that $g(x)$ is in the domain of f.

Example: Suppose $f(x) = \sqrt{x}$ and $g(x) = 2x + 1$. Then $f(\ g(x)\ ) = f(2x + 1) = \sqrt{2x + 1}$.

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Example: Suppose $g \equiv \{(1, 2), (3, 4), (5, 6)\}$ and $f \equiv \{(2, 8), (4, 9), (1, 1)\}$. Find $f \circ g$.

Solution: Since $g(1) = 2$ and $f(2) = 8$, then $f(g(1)) = 8$, and $(1, 8)$ is an ordered pair in $f \circ g$. Also since $g(3) = 4$ and $f(4) = 9$, then $f(g(3)) = 9$, and $(3, 9)$ is an ordered pair in $f \circ g$. Now $g(5) = 6$ but 6 is not in the domain of f. So there are only two ordered pairs in $f \circ g$, namely $f \circ g \equiv \{(1, 8), (3, 9)\}$

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Example: Suppose $g \equiv \{(1,2),(3,4),(5,6)\}$ and $f \equiv \{(2,8),(4,9),(1,1)\}$. Find $f \circ g$.

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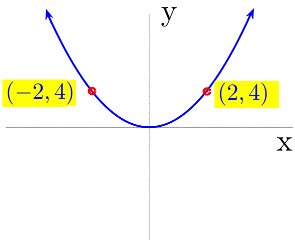
Comment: the domain of g is $\{1,3,5\}$ while the domain of $f \circ g$ is $\{1,3\}$. **In order to find the domain of $f \circ g$ we remove from the domain of g any number x such that $g(x)$ is not in the domain of f.**

One to one functions have inverses!

A function f with domain D and range R is a <u>one to one function</u> if *either* of the following equivalent conditions is satisfied:

Whenever $x_1 \neq x_2$ in D, then $f(x_1) \neq f(x_2)$ in R.

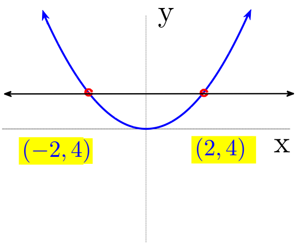
Whenever $f(x_1) = f(x_2)$ in R, then $x_1 = x_2$ in D.



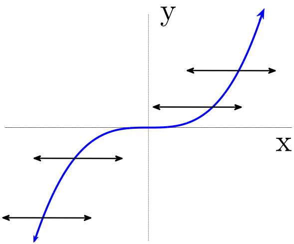
Example: $f(x) = x^2$ is *NOT* a one to one function since for $x_1 = -2$ and $x_2 = 2$, it is true that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = 4$.

The Horizontal Line Test

A function f is one to one if and only if every horizontal line intersects the graph of f in at most one point.



$f(x) = x^2$
is not one to one



but $f(x) = x^3$
is one to one.

Inverse Function

Suppose f is a one to one function, with domain D and range R. The inverse function of f is the function denoted f^{-1} with domain R and range D provided that

$$f^{-1}(f(x)) = x$$

Note: A function has an inverse (function) only when it is *one to one*.

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CAUTION: $f^{-1}(x) \neq f(x)^{-1}$

- $f^{-1}(x)$ is notation for the function inverse of a one to one function f
- $f(x)^{-1} = (f(x))^{-1} = \dfrac{1}{f(x)}$ is the multiplicative inverse of the number $f(x)$.

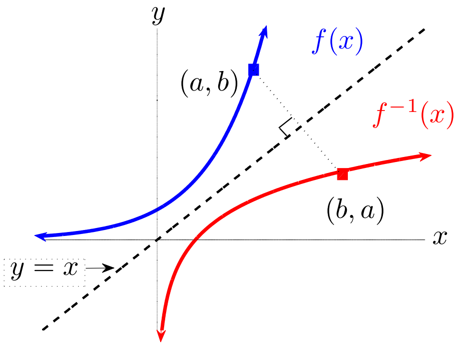
Example: Suppose f is one-to-one and $f(-9) = 15$, then $f^{-1}(15) = -9$ and $(f(-9))^{-1} = 1/15$

Properties of Inverse Functions

Suppose that f is a one to one function with domain D and range R. Then

- The inverse function f^{-1} is unique.
- The domain of f^{-1} is the range of f.
- The range of f^{-1} is the domain of f.
- The statement $f(x) = y$ is equivalent to $f^{-1}(y) = x$

Note: The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$. For every point (a, b) on the graph of $f(x)$ there is a corresponding point (b, a) on the graph of $f^{-1}(x)$.



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Inverse Function

How to find the inverse of a one to one function:

1. Replace $f(x)$ with y. Then interchange x and y.
2. Solve the resulting equation for y.
3. Replace y with $f^{-1}(x)$.