

Polynomial Division

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Dividing Polynomials

Simple Case: Division by a Monomial

Example: Divide $\frac{6x^3 - 9x^2 + 12x}{3x}$

Solution

$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

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$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

$$= \frac{6x^3}{3x} - \frac{9x^2}{3x} + \frac{12x}{3x} = 2x^2 - 3x + 4$$

Try This One! Divide $\frac{27x^4y^7 - 81x^5y^3}{-9x^3y^2}$ to lowest terms.

Dividing Polynomials

Procedure

Whenever the denominator is not a monomial, or a factor of the numerator, or if the numerator is not factorable, the previous method won't work. So, instead we use long division of polynomials, a method similar to long division of whole numbers.

Theorem (Division Algorithm):

Suppose D and P are polynomial expressions of variable x , with $D \neq 0$, and suppose that D is less than the degree of P . Then there exist unique polynomials Q and R , where R is either 0 or has degree less than the degree of D , such that

$$P = Q \cdot D + R \text{ or, equivalently } \frac{P}{D} = Q + \frac{R}{D}$$

In words, we have

$$\text{dividend} = (\text{quotient}) \cdot (\text{divisor}) + \text{remainder}$$

or

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Long Division of Polynomials

Dividing $\frac{x^3 + 2x^2 - x - 2}{x - 1}$ is equivalent to the long division problem and solution:

Divisor, D(x) \rightarrow $x - 1$

| | | |
|----------------------|--|--------------------------------------|
| $x^2 + 3x + 2$ | | $\leftarrow \text{Quotient, } Q(x)$ |
| $x^3 + 2x^2 - x - 2$ | | $\leftarrow \text{Dividend, } P(x)$ |
| $x^3 - x^2$ | | |
| $3x^2 - x - 2$ | | |
| $3x^2 - 3x$ | | |
| $2x - 2$ | | |
| $2x - 2$ | | |
| 0 | | $\leftarrow \text{Remainder, } R(x)$ |

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Set the problem up for long division. Write the dividend in descending order and insert zero placeholders for any missing polynomial terms, if necessary.

$$x - 1 \overline{)x^3 + 2x^2 - x - 2}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor.

$$x - 1 \overline{)x^3 + 2x^2 - x - 2}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor.

That is, divide $\frac{x^3}{x}$

$$\begin{array}{r} x - 1 \quad \overline{)x^3 + 2x^2 - x - 2} \\ \downarrow \quad \downarrow \\ x^3 \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor.

That is, divide $\frac{x^3}{x} = x^2$. Write this above the x^2 term of the dividend.

$$\begin{array}{r} & & x^2 \\ x - 1 \quad | & x^3 & + & 2x^2 & - & x & - 2 \\ \swarrow & \searrow & & & & & \\ \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $x^2 \cdot (x - 1)$ and list the result below $x^3 + 2x^2 - x - 2$

$$\begin{array}{c} & & x^2 \\ & \swarrow & \curvearrowleft x \\ x - 1 & \overline{)x^3 + 2x^2 - x - 2} \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $x^2 \cdot (x - 1) = x^3 - x^2$

$$\begin{array}{r} & & x^2 \\ x - 1 & \overline{)x^3 + 2x^2 - x - 2} \\ & x^3 - & x^2 \\ \hline & & \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD

$$\begin{array}{r} & x^2 \\ x - 1 \Big) & \overline{x^3 + 2x^2 - x - 2} \\ & -x^3 + x^2 \\ \hline & 0 + 3x^2 \end{array}$$

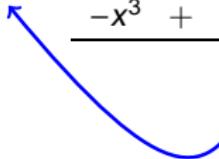
Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Bring $-x - 2$ down.

$$\begin{array}{r} x^2 \\ x - 1 \left[\begin{array}{cccc} x^3 & + & 2x^2 & - & x & - 2 \\ -x^3 & + & x^2 & & & \\ \hline 3x^2 & - & x & - 2 & & \end{array} \right] \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat the process using $3x^2 - x - 2$ as the dividend. Add $\frac{3x^2}{x} = 3x$ to the quotient.

$$\begin{array}{r} x^2 + 3x \\ x - 1 \overline{) x^3 + 2x^2 - x - 2} \\ -x^3 + x^2 \\ \hline 3x^2 - x - 2 \end{array}$$


Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $3x \cdot (x - 1)$ and list the result below $3x^2 - x - 2$.

A handwritten diagram of polynomial long division. The divisor $x - 1$ is written to the left of a bracket under the dividend $x^3 + 2x^2 - x - 2$. A red curved arrow points from the term x in $x - 1$ up to the term x^3 in the dividend. Above the dividend, the quotient terms x^2 , $+$, and $3x$ are written above the line. Below the dividend, the first term of the partial product $-x^3$ is written, followed by a plus sign and the term x^2 . A horizontal line separates the partial product from the remainder. The remainder $3x^2 - x - 2$ is written below the line.

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $3x \cdot (x - 1) = 3x^2 - 3x$

A handwritten diagram of polynomial long division. The divisor is $x - 1$, written vertically on the left. The dividend is $x^3 + 2x^2 - x - 2$, written horizontally above the division bar. Inside the division bar, the first term x^3 is divided by x to get x^2 . The term $2x^2$ is multiplied by $x - 1$ to get $-x^3 + x^2$, which is then subtracted from the dividend. The result is $3x^2 - x - 2$. This result is then multiplied by $x - 1$ again to get $3x^2 - 3x$, which is also subtracted from the dividend. A red curved arrow points from the term $3x$ in the quotient to the term $3x^2$ in the remainder. A blue curved arrow points from the term $3x$ in the remainder to the term $3x^2$ in the next step.

$$\begin{array}{r} x^2 + 3x \\ \hline x - 1 \left| \begin{array}{r} x^3 + 2x^2 - x - 2 \\ -x^3 + x^2 \\ \hline 3x^2 - x - 2 \\ 3x^2 - 3x \end{array} \right. \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD.

$$\begin{array}{r} x^2 + 3x \\ x - 1 \overline{)x^3 + 2x^2 - x - 2} \\ -x^3 + x^2 \\ \hline 3x^2 - x - 2 \\ -3x^2 + 3x \\ \hline 0 + 2x \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Bring -2 down.

$$\begin{array}{r} & x^2 & + & 3x \\ x - 1 & \overline{)x^3 & + & 2x^2 & - & x & - 2} \\ & -x^3 & + & x^2 \\ \hline & & 3x^2 & - & x & \textcolor{red}{- 2} \\ & & -3x^2 & + & 3x \\ \hline & & & & 2x & \textcolor{red}{- 2} \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat the process using $2x - 2$ as the dividend. Add $\frac{2x}{x} = 2$ to the quo-

$$\begin{array}{r} x^2 + 3x + 2 \\ \hline x - 1 \left[\begin{array}{r} x^3 + 2x^2 - x - 2 \\ -x^3 + x^2 \\ \hline 3x^2 - x - 2 \\ -3x^2 + 3x \\ \hline 2x - 2 \end{array} \right] \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $2 \cdot (x - 1)$ and list the result below $2x - 2$.

The diagram shows the long division of $x^3 + 2x^2 - x - 2$ by $x - 1$. A red curved arrow points from the term $x^2 + 3x + 2$ to the term $x^3 + 2x^2 - x - 2$. A green arrow points from the divisor $x - 1$ to the first term of the dividend x^3 . The quotient is $x^2 + 3x + 2$, and the remainder is $2x - 2$.

$$\begin{array}{r} x^2 + 3x + 2 \\ x - 1 \overline{) x^3 + 2x^2 - x - 2} \\ -x^3 + x^2 \\ \hline 3x^2 - x - 2 \\ -3x^2 + 3x \\ \hline 2x - 2 \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $2 \cdot (x - 1) = 2x - 2$

$$\begin{array}{r} x^2 + 3x + 2 \\ x - 1 \overline{)x^3 + 2x^2 - x - 2} \\ -x^3 + x^2 \\ \hline 3x^2 - x - 2 \\ -3x^2 + 3x \\ \hline 2x - 2 \\ 2x - 2 \\ \hline \end{array}$$

Example: Divide $\frac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD

$$\begin{array}{r} x^2 + 3x + 2 \\ x - 1 \left[\begin{array}{r} x^3 + 2x^2 - x - 2 \\ -x^3 + x^2 \\ \hline 3x^2 - x - 2 \\ -3x^2 + 3x \\ \hline 2x - 2 \\ -2x + 2 \\ \hline 0 \end{array} \right] \end{array}$$

Hence the remainder, $R(x) = 0$, and

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

How does one know when the long division process is finished?

It is not always the case that the remainder will be zero when dividing two polynomials. So, how does one know when the long division process of polynomials is over? **When the degree of the remainder is less than the degree of the divisor.**

Recall that the degree of a polynomial expression is the largest power of x that occurs amongst the terms in the expression.

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$$\begin{array}{r} x^2 + 3x + 2 \\ x - 1 \left[\begin{array}{r} x^3 + 2x^2 - x - 2 \\ x^3 - x^2 \\ \hline 3x^2 - x - 2 \\ 3x^2 - 3x \\ \hline 2x - 2 \\ 2x - 2 \\ \hline 0 \end{array} \right] \end{array}$$