Chapter 3

Professor Tim Busken

Mathematics Department

February 17, 2014

Table of Contents

Definition

A **term** is either a single number or variable, or the product or quotient of several numbers or variables separated from another term by a plus or minus sign in an overall expression.

Definition

A **term** is either a single number or variable, or the product or quotient of several numbers or variables separated from another term by a plus or minus sign in an overall expression.

For example, the following algebraic expression

$$100 + 3x + 5yz^2w^3 - \frac{2}{3}x$$

has terms $100, 3x$, $5yz^2w^3$, and $\frac{2}{3}x$.

Definition

A **term** is either a single number or variable, or the product or quotient of several numbers or variables separated from another term by a plus or minus sign in an overall expression.

For example, the following algebraic expression

$$100 + 3x + 5yz^2w^3 - \frac{2}{3}x$$

has terms $100, 3x$, $5yz^2w^3$, and $\frac{2}{3}x$.

Definition

The numerical factor of a term is a **coefficient**.

Definition

A **term** is either a single number or variable, or the product or quotient of several numbers or variables separated from another term by a plus or minus sign in an overall expression.

For example, the following algebraic expression

$$100 + 3x + 5yz^2w^3 - \frac{2}{3}x$$

has terms $100, 3x$, $5yz^2w^3$, and $\frac{2}{3}x$.

Definition

The numerical factor of a term is a **coefficient**.

For example, the aforementioned terms have coefficients $100, 3$, 5, and $\frac{2}{3}$.

Definition

A **term** is either a single number or variable, or the product or quotient of several numbers or variables separated from another term by a plus or minus sign in an overall expression.

For example, the following algebraic expression

$$100 + 3x + 5yz^2w^3 - \frac{2}{3}x$$

has terms $100, 3x$, $5yz^2w^3$, and $\frac{2}{3}x$.

Definition

The numerical factor of a term is a **coefficient**.

For example, the aforementioned terms have coefficients $100, 3$, 5, and $\frac{2}{3}$.

Definition

A **constant** is a single number, such as 8 or 9.

Definition

A **monomial function** has the form

$$p(x) = ax^n,$$

where a is a constant that is any real number, x is a variable, and n is a whole number $(0, 1, 2, \ldots)$.

Definition

A **monomial function** has the form

$$p(x) = ax^n,$$

where a is a constant that is any real number, x is a variable, and n is a whole number $(0, 1, 2, \ldots)$.

For instance,

$$p(x) = 3, \qquad p(x) = 5x, \qquad p(x) = 7x^4, \qquad \text{and} \qquad p(x) = 9x^{200}$$

are all examples of monomial functions.

Definition

We call n the **degree** of the monomial function.

Definition

We call n the **degree** of the monomial function.

The degree of a nonzero constant is zero. Because $0 = 0x = 0x^2 = 0x^3 = \ldots$, we cannot assign a degree to the $p(x) = 0$. Therefore, we say 0 has no degree.

Monomial	Coefficient	Degree
3	3	0
$-5x^2$	-5	2
x^7	1	7
0	0	no degree

Definition

We call n the **degree** of the monomial function.

The degree of a nonzero constant is zero. Because $0 = 0x = 0x^2 = 0x^3 = \ldots$, we cannot assign a degree to the $p(x) = 0$. Therefore, we say 0 has no degree.

Monomial	Coefficient	Degree
3	3	0
$-5x^2$	-5	2
x^7	1	7
0	0	no degree

- $p(x) = 4x^{-3}$ is not a monomial function because the exponent of the variable, x, is -3 and -3 is not a whole number.

Definition

We call n the **degree** of the monomial function.

The degree of a nonzero constant is zero. Because $0 = 0x = 0x^2 = 0x^3 = \ldots$, we cannot assign a degree to the $p(x) = 0$. Therefore, we say 0 has no degree.

Monomial	Coefficient	Degree
3	3	0
$-5x^2$	-5	2
x^7	1	7
0	0	no degree

- $p(x) = 4x^{-3}$ is not a monomial function because the exponent of the variable, x, is -3 and -3 is not a whole number.

- $p(x) = 2x^{1/3}$ is not a monomial function because the exponent of the variable is $1/3$, and $1/3$ is not a whole number.

A polynomial function of degree n is a function of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer and $a_n \neq 0$.

- The numbers $a_n, a_{n-1}, \ldots, a_3, a_2, a_1, a_0$ are the *COEFFICIENTS* of the polynomial.
- a_0 is called the *CONSTANT TERM*.
- $a_n x^n$ is called the *LEADING TERM* of the polynomial.
- a_n is called the *LEADING COEFFICIENT* of the polynomial.
- n is called the *DEGREE* of the polynomial.

Definition

A **polynomial** is a monomial or a sum of monomials.

$f(x) = 11$	monomial
$f(x) = 3x^4$	monomial function
$f(x) = 2x^2 + 1$	binomial function
$f(x) = 5x^3 + x - 1$	trinomial function
$f(x) = x^{1/2} + 5$	is not a polynomial function
$f(x) = \sqrt[5]{x + 5}$	is not a polynomial function
$f(x) = \dfrac{1}{x - 1}$	is not a polynomial function

Definition

The **degree of polynomial function** is the degree of the leading term (the term which has x raised to the largest power).

Polynomial Function	Degree	Leading Term	Leading Coefficient	Constant term
$f(x) = -2x^4 - 3x - 5$	4	$-2x^4$	-2	-5
$f(x) = x^5 - 3x^6 - 10x - 4$	6	$-3x^6$	-3	-4
$f(x) = 5x^{10} - 8x^3 - 10x + 5$	10	$5x^{10}$	5	5
$f(x) = 17x + 4$	1	$17x$	17	4
$f(x) = 24$	0	24	24	24

Quadratic Functions

Suppose a, b and c are real numbers and that a is not zero. A quadratic function is a function having the form

$$g(x) = ax^2 + bx + c$$

A quadratic function is a 2nd-degree polynomial.

(The condition $a \neq 0$ in the definition ensures that the function is second degree polynomial, and not a first degree polynomial.)

Useful Terminology

The **square of a number** is the number times itself.
For instance, the square of 4 is 16 because 4^2 or $4 \cdot 4 = 16$.

A **perfect square** is a number that is the square of an integer. In other words, we can take an integer and square it, and obtain another number. This number is called a perfect square.
For instance, 25 is a perfect square because $25 = 5 \cdot 5 = 5^2$.

Example: Recall that a "trinomial" is a polynomial which has 3 terms. $x^2 + 6x + 9$ is called a **perfect-square trinomial** because it is a trinomial which factors like a *perfect square* number.

$$\begin{aligned} x^2 + 6x + 9 &= (x+3) \cdot (x+3) \\ &= (x+3)^2 \end{aligned}$$

A **perfect-square trinomial** is a polynomial with the properties:

1. two of the three terms must be squares, such as A^2 and B^2
2. neither A^2 nor B^2 is being subtracted
3. the remaining term must be $2 \cdot A \cdot B$ or its opposite $-2 \cdot A \cdot B$.

We add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to **complete the square**. That is, if we add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$, the sum $x^2 + bx + \left(\frac{b}{2}\right)^2$ is a trinomial that factors like a perfect-square since

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

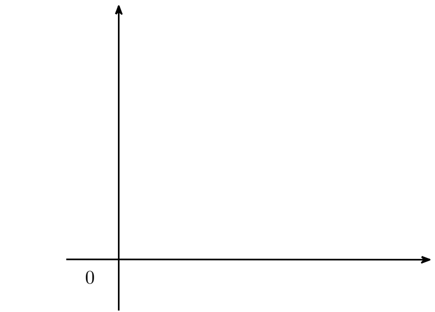
Example: Rewrite $g(x) = x^2 + 8x$ in an equivalent form by completing the square. The result will allow us to write $g(x)$ in terms of $f(x) = x^2$ so we can use translations theory to graph g.

Standard Form

Suppose h and k represent real numbers. Then $g(x) = ax^2 + bx + c$ can be expressed in **standard form**

$$g(x) = a\,(x - h)^2 + k$$

by completing the square.



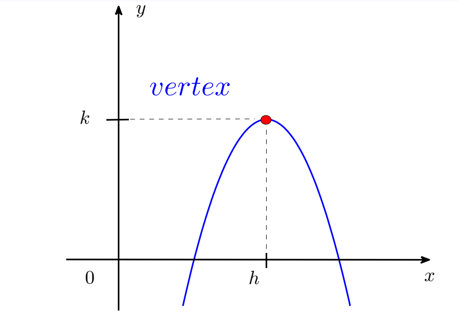
Standard Form

Suppose h and k represent real numbers. Then $g(x) = ax^2 + bx + c$ can be expressed in **standard form**

$$g(x) = a\,(x-h)^2 + k$$

by completing the square.

- The graph of g is a parabola with **vertex** $(x, y) = (h, k)$



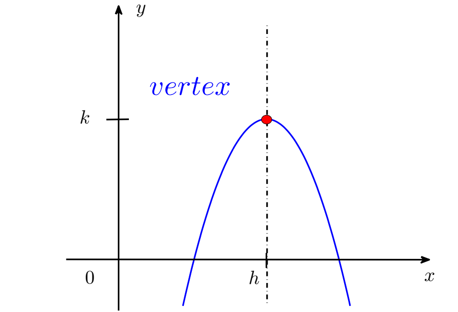
Standard Form

Suppose h and k represent real numbers. Then $g(x) = ax^2 + bx + c$ can be expressed in **standard form**

$$g(x) = a\,(x - h)^2 + k$$

by completing the square.

- The graph of g is a parabola with **vertex** $(x, y) = (h, k)$
- and axis of symmetry (A.O.S.) $x = h$.



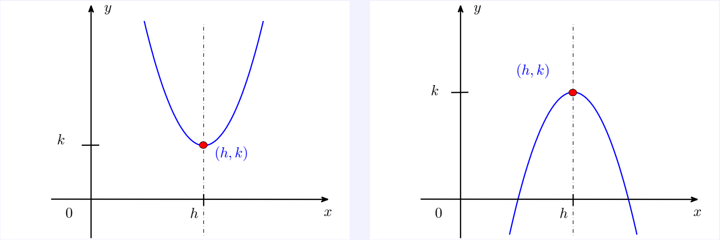
Standard Form

Suppose h and k represent real numbers. Then $g(x) = ax^2 + bx + c$ can be expressed in **standard form**

$$g(x) = a(x-h)^2 + k$$

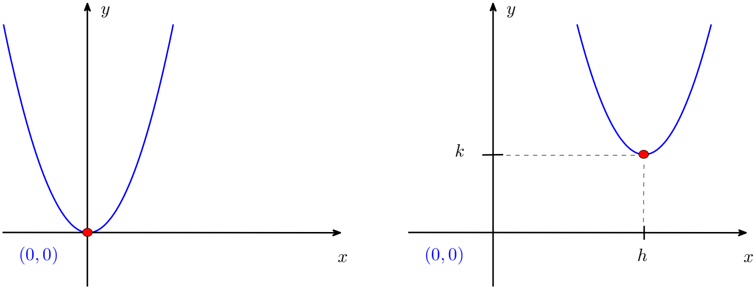
by completing the square.

- The graph of g is a parabola with **vertex** $(x, y) = (h, k)$
- and axis of symmetry (A.O.S.) $x = h$.
- The parabola opens upward if $a > 0$ or downward if $a < 0$.



TRANSLATION OF QUADRATIC GRAPHS

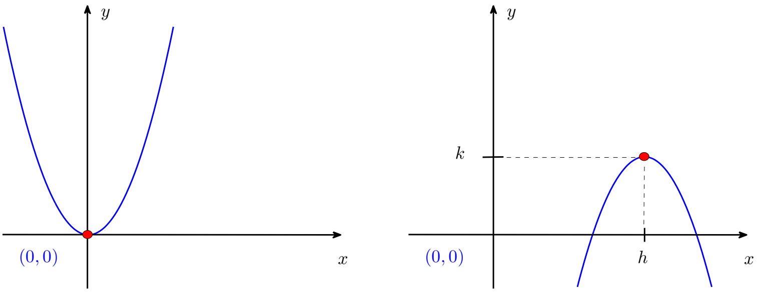
The standard form of a quadratic, $g(x) = a \cdot f(x-h) + k$, is a **transformation** of $f(x) = x^2$.



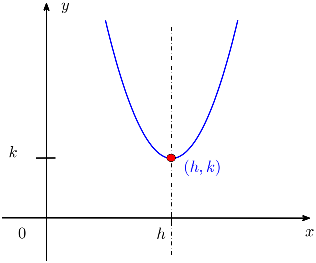
- The vertex point $(0, 0)$ on the graph of $f(x) = x^2$ is translated to (h, k) on the graph of $g(x) = a(x-h)^2 + k$.
- Since $(0, 0)$ is the lowest point on the graph of $f(x) = x^2$, the lowest point on any parabola that opens upward is (h, k).

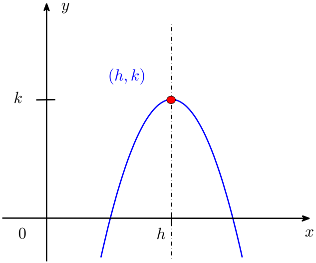
TRANSLATION OF QUADRATIC GRAPHS

The standard form of a quadratic, $g(x) = a \cdot f(x-h) + k$, is a **transformation** of $f(x) = x^2$.



- The vertex point $(0,0)$ on the graph of $f(x) = x^2$ is translated to (h,k) on the graph of $g(x) = a(x-h)^2 + k$.
- The maximum point on any parabola that opens downward is (h,k).
- Translations GeoGebra Worksheet





- The domain of any quadratic function is the set of all real numbers $x \in (-\infty, \infty)$.

- The range of a quadratic function is determined from the second coordinate of the vertex, k.

- If $a > 0$, the range is $y \in [k, \infty)$ and k is called the minimum value of the function. The function is decreasing for $x \in (-\infty, h]$ and increasing for $x \in [h, \infty)$.

- If $a < 0$, the range is $y \in (-\infty, k]$ and k is called the maximum value of the function. The function is increasing for $x \in (-\infty, h]$ and decreasing for $x \in [h, \infty)$.

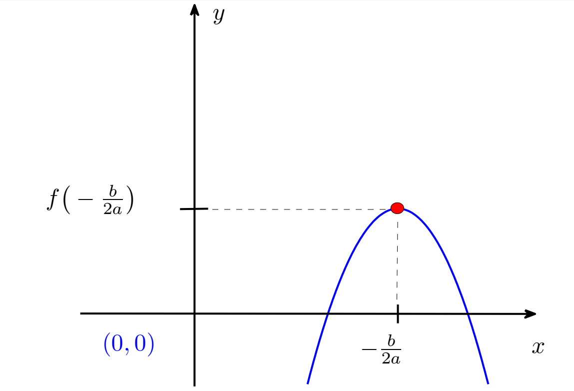
Quadratic Functions

A quadratic function of the form

$$f(x) = ax^2 + bx + c$$

has vertex point

$$(x, y) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$



Proof:

We will complete the square and write $f(x) = ax^2 + bx + c$ in vertex form.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\cdot\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\cdot\left(x^2 + \frac{b}{a}x + \underline{\qquad}\right) + \left(c - \underline{\qquad}\right) \\ &= a\cdot\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \left(c - \frac{b^2}{4a}\right) \end{aligned}$$

Notice that if $b^2/(4a^2)$ is added inside of the parenthesis, then, because of the factor a on the outside, we have actually added $b^2/(4a)$ to $f(x)$. So, we need to compensate by subtracting $b^2/(4a)$. Since the expression in the first set of parenthesis is in the form of a perfect-square trinomial, we can write $f(x)$ as

$$f(x) = a\cdot\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

We know from translating the graph of $y = x^2$, that this is the equation of a parabola with vertex (h, k) where $h = -b/(2a)$ and $k = c - b^2/(4a)$. It is easy to show that $f(-b/2a) = c - b^2/(4a)$.

$\therefore$, $f(x) = ax^2 + bx + c$ has vertex point $(x, y) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Proof of the quadratic formula:

We already know that $f(x) = ax^2 + bx + c$ has the equivalent vertex form

$$f(x) = a \cdot \left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

We set $f(x)$ equal to zero and solve for x to find the x-coordinates of the x intercepts of the parabola, if they exist.

$$0 = a \cdot \left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{Replace } f(x) \text{ with zero}$$

$$-\left(c - \frac{b^2}{4a}\right) = a \cdot \left(x + \frac{b}{2a}\right)^2 \qquad \text{Subtract } \left(c - \frac{b^2}{4a}\right) \text{ from both sides}$$

$$\frac{b^2}{4a} - c = a \cdot \left(x + \frac{b}{2a}\right)^2 \qquad \text{Distribute } -1 \text{ and use commutivity}$$

$$\frac{b^2 - 4ac}{4a} = a \cdot \left(x + \frac{b}{2a}\right)^2 \qquad \text{Subtract } \frac{b^2}{4a} - c.$$

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2 \qquad \text{Divide both sides by } a \ (\neq 0)$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Apply the Square Root Theorem}$$

Proof Continued

We already know that

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$ Use property $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$ Since $\sqrt{4a^2} = 2 \cdot |a| = \pm 2a$.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ Add the opposite of $\frac{b}{2a}$ to both sides.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Add fractions with like denominators.

But, this is the Quadratic Formula—which is what we wanted to show.

The expression underneath the square root, $b^2 - 4ac$, has a special name, because it determines the number of real solutions of $f(x) = 0$, and the number of x-intercepts of f's graph.

The Discriminant

Definition

$b^2 - 4ac$ is called the *discriminant*, because it determines the number of x-intercepts of $f(x) = ax^2 + bx + c$.

Value of $b^2 - 4ac$	# of real solns	Number of x intercepts
positive	2	2
zero	1	1
negative	0	0

The graph of a quadratic function $f(x)$ is completely above (or below) the x axis whenever $f(x)$ has a negative discriminant.

Example: Solve $x^2 + 3x - 1 > -3$

Example: Solve $x^2 + 3x - 1 > -3$

Test Point Method

(Step 1) Get a zero on one side of the inequality.

$$x^2 + 3x - 1 > -3$$

Example: Solve $x^2 + 3x - 1 > -3$

Test Point Method

(Step 1) Get a zero on one side of the inequality.

$$x^2 + 3x - 1 > -3$$

$$x^2 + 3x - 1 + 3 > -3 + 3$$

$$x^2 + 3x + 2 > 0$$

Example: Solve $x^2 + 3x - 1 > -3$

Test Point Method

(Step 1) Get a zero on one side of the inequality.

$$x^2 + 3x - 1 > -3$$

$$x^2 + 3x - 1 + 3 > -3 + 3$$

$$x^2 + 3x + 2 > 0$$

The given inequality problem is equivalent to the finding the solution set to $f(x) > 0$ where $f(x) = x^2 + 3x + 2$.

Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

(**Step 2**) Factor $f(x)$

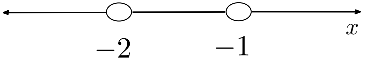
$$(x+2)(x+1) > 0$$

Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

(**Step 2**) Factor $f(x)$

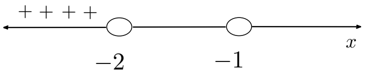
$$(x+2)(x+1) > 0$$

(**Step 3**) Locate the x-intercepts of f on a coordinate line. This subdivides the number line into three subintervals: $(-\infty, -2) \bigcup (-2, -1) \bigcup (-1, \infty)$.



Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

(**Step 4**) Select a test value of x from the leftmost subinterval, and substitute it into $f(x)$ to determine the sign ($+$ or $-$) of f for that interval. Mark the sign above the coordinate line.



For $x \in (-\infty, -2)$, we select test value $x = -3$ and find that

$$f(-3) = 2 > 0$$

$\Longrightarrow f(x) > 0$ for all $x \in (-\infty, -2)$.

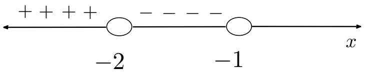
So we mark the interval on the x axis graph with plus signs to indicate this.

Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

(**Step 4**) For $x \in (-2, -1)$, we select test value $x = -\frac{3}{2}$ and find that

$$f\left(-\frac{3}{2}\right) = \left(-\frac{3}{2} + 2\right)\cdot\left(-\frac{3}{2} + 1\right) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) < 0$$

$\Longrightarrow f(x) < 0$ for all $x \in (-2, -1)$ so we mark the interval on the x axis graph with negative signs.

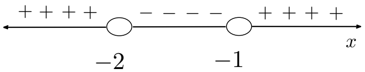


Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

(**Step 4**) For $x \in (-1, \infty)$, we use $x = 0$ and find that

$$f(0) = (0+2)\cdot(0+1) = (-1)\cdot(-2) = 2 > 0$$

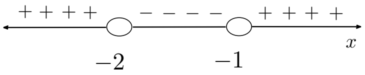
$\implies f(x) > 0$ for all $x \in (-1, \infty)$ so we mark the interval on the x axis graph with plus signs.



Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

(**Step 5**) Now check to see which x interval(s) satisfy the inequality.

For this problem, we need all values of x which make $y = x^2 + 3x + 2$ positive, or above the x axis: $\boxed{x \in (-\infty, -2) \bigcup (-1, \infty)}$.



Alternatively, had the inequality been given as a less than symbol, we would instead take $x \in (-2, -1)$ as the soln set.

Example: Solve $f(x) > 0$ where $f(x) = x^2 + 3x + 2$

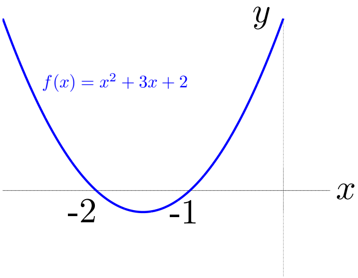
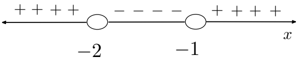


Figure : Soln Set: $x \in (-\infty, -2) \bigcup (-1, \infty)$



Dividing Polynomials

Simple Case: Division by a Monomial

Example: Divide $\dfrac{6x^3 - 9x^2 + 12x}{3x}$

Solution

$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x)$$

Dividing Polynomials

Simple Case: Division by a Monomial

Example: Divide $\dfrac{6x^3 - 9x^2 + 12x}{3x}$

Solution

$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

Dividing Polynomials

Simple Case: Division by a Monomial

Example: Divide $\dfrac{6x^3 - 9x^2 + 12x}{3x}$

Solution

$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

$$= \frac{6x^3}{3x} - \frac{9x^2}{3x} + \frac{12x}{3x} = 2x^2 - 3x + 4$$

Try This One! Divide $\dfrac{27x^4y^7 - 81x^5y^3}{-9x^3y^2}$ to lowest terms.

Dividing Polynomials

Procedure

Whenever the denominator is not a monomial, or a factor of the numerator, or if the numerator is not factorable, the previous method won't work. So, instead we use long division of polynomials, a method similar to long division of whole numbers.

Theorem (Division Algorithm:)

Suppose $D(x)$ and $P(x)$ are polynomial functions of x with $D(x) \neq 0$, and suppose that $D(x)$ is less than the degree of $P(x)$. Then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or has degree less than the degree of $D(x)$, such that

$$P(x) = Q(x) \cdot D(x) + R(x)$$

In words, we have

$$\text{dividend} = (\text{quotient})\cdot(\text{divisor}) + \text{remainder}$$

or

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Long Division of Polynomials

Dividing $\dfrac{x^3+2x^2-x-2}{x-1}$ is equivalent to the long division problem and solution:

$$
\begin{array}{rr|rrrrrrl}
 & & x^2 & + & 3x & + & 2 & & \leftarrow \text{Quotient, } Q(x) \\
\cline{3-8}
\text{Divisor, } D(x) \twoheadrightarrow & x-1 & x^3 & + & 2x^2 & - & x & -2 & \leftarrow \text{Dividend, } P(x) \\
 & & x^3 & - & x^2 & & & & \\
\cline{3-8}
 & & & & 3x^2 & - & x & -2 & \\
 & & & & 3x^2 & - & 3x & & \\
\cline{5-8}
 & & & & & & 2x & -2 & \\
 & & & & & & 2x & -2 & \\
\cline{7-8}
 & & & & & & & 0 & \leftarrow \text{Remainder, } R(x)
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Set the problem up for long division. Write the dividend in descending order and insert zero placeholders for any missing polynomial terms, if necessary.

$$x - 1 \overline{\big)\; x^3 \;+\; 2x^2 \;-\; x \;-\; 2}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor.

$$x - 1 \,\overline{\big)\, x^3 + 2x^2 - x - 2}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor. That is, divide $\dfrac{x^3}{x}$

$$x - 1 \,\overline{\big)\, x^3 + 2x^2 - x - 2}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor. That is, divide $\dfrac{x^3}{x} = x^2$. Write this above the x^2 term of the dividend.

$$\begin{array}{r|l} & \quad x^2 \\ \hline x - 1 & x^3 + 2x^2 - x - 2 \end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $x^2 \cdot (x - 1)$ and list the result below $x^3 + 2x^2 - x - 2$

$$\begin{array}{r|l} & x^2 \\ \hline x - 1 & x^3 + 2x^2 - x - 2 \end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $x^2 \cdot (x - 1) = x^3 - x^2$

$$x^2$$

$$x - 1 \overline{) \; x^3 + 2x^2 - x - 2}$$

$$x^3 - x^2$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD

$$
\begin{array}{r|rrrr}
 & x^2 & & & \\
\hline
x - 1 & x^3 + & 2x^2 & - x & - 2 \\
 & -x^3 + & x^2 & & \\
\hline
 & 0 + & 3x^2 & &
\end{array}
$$

Example: Divide $\dfrac{x^3+2x^2-x-2}{x-1}$

Solution: Bring $-x-2$ down.

$$
\begin{array}{r|rrrr}
 & x^2 & & & \\
\hline
x-1 & x^3 + & 2x^2 & - x & - 2 \\
 & -x^3 + & x^2 & & \\
\hline
 & & 3x^2 & - x & - 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat the process using $3x^2 - x - 2$ as the dividend. Add $\dfrac{3x^2}{x} = 3x$ to the quotient.

$$
\begin{array}{r|rrrr}
 & x^2 & + \; 3x & & \\
\hline
x - 1 & x^3 + & 2x^2 & - \; x & - \; 2 \\
 & -x^3 + & x^2 & & \\
\hline
 & & 3x^2 & - \; x & - \; 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $3x \cdot (x - 1)$ and list the result below $3x^2 - x - 2$.

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 3x & & \\
\cline{2-8}
x - 1 & x^3 & + & 2x^2 & - & x & - & 2 \\
 & -x^3 & + & x^2 & & & & \\
\cline{2-8}
 & & & 3x^2 & - & x & - & 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $3x \cdot (x - 1) = 3x^2 - 3x$

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 3x & & \\
\hline
x - 1 & x^3 & + & 2x^2 & - & x & - & 2 \\
 & -x^3 & + & x^2 & & & & \\
\hline
 & & & 3x^2 & - & x & - & 2 \\
 & & & 3x^2 & - & 3x & &
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD.

$$\begin{array}{r|rrrr} & x^2 & +\ 3x & & \\ \cline{2-5} x-1 & x^3 + & 2x^2 & -\ x & -\ 2 \\ & -x^3 + & x^2 & & \\ \cline{2-5} & & 3x^2 & -\ x & -\ 2 \\ & & -3x^2 & +\ 3x & \\ \cline{3-5} & & 0 & +\ 2x & \end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Bring -2 down.

$$\begin{array}{r|llll}
 & x^2 & + \; 3x & & \\ \hline
x - 1 & x^3 + & 2x^2 & - \; x & - \; 2 \\
 & -x^3 + & x^2 & & \\ \hline
 & & 3x^2 & - \; x & - \; 2 \\
 & & -3x^2 & + \; 3x & \\ \hline
 & & & 2x & - \; 2
\end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat the process using $2x - 2$ as the dividend. Add $\dfrac{2x}{x} = 2$ to the quotient.

$$
\begin{array}{r|rrrr}
 & x^2 & + \; 3x & + \; 2 & \\
\hline
x - 1 & x^3 + & 2x^2 & - \; x & - \; 2 \\
 & -x^3 + & x^2 & & \\
\hline
 & & 3x^2 & - \; x & - \; 2 \\
 & & -3x^2 & + \; 3x & \\
\hline
 & & & 2x & - \; 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $2 \cdot (x - 1)$ and list the result below $2x - 2$.

$$
\begin{array}{r|rrrr}
 & & x^2 & +\ 3x & +\ 2 \\
\hline
x - 1 & x^3 & +\ 2x^2 & -\ x & -\ 2 \\
 & -x^3 & +\ x^2 & & \\
\hline
 & & 3x^2 & -\ x & -\ 2 \\
 & & -3x^2 & +\ 3x & \\
\hline
 & & & 2x & -\ 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $2 \cdot (x - 1) = 2x - 2$

$$
\begin{array}{r|rrrr}
 & & x^2 & +\ 3x & +\ 2 \\
\hline
x - 1 & x^3 & +\ 2x^2 & -\ x & -\ 2 \\
 & -x^3 & +\ x^2 & & \\
\hline
 & & 3x^2 & -\ x & -\ 2 \\
 & & -3x^2 & +\ 3x & \\
\hline
 & & & 2x & -\ 2 \\
 & & & 2x & -\ 2
\end{array}
$$

Example: Divide $\dfrac{x^3+2x^2-x-2}{x-1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD

$$
\begin{array}{r|rrrr}
 & x^2 & + 3x & + 2 & \\
\hline
x-1 & x^3 + & 2x^2 & - x & - 2 \\
 & -x^3 + & x^2 & & \\
\hline
 & & 3x^2 & - x & - 2 \\
 & & -3x^2 & + 3x & \\
\hline
 & & & 2x & - 2 \\
 & & & -2x & +2 \\
\hline
 & & & & 0
\end{array}
$$

Hence the remainder, $R(x) = 0$, and

$$\frac{x^3+2x^2-x-2}{x-1} = x^2+3x+2$$

The remainder is not always zero when dividing two polynomials. So, how does one know when the long division process is finished?

When the degree of the remainder is less than the degree of the divisor.

Recall that the degree of a polynomial expression is the largest power of x that occurs amongst the terms in the expression.

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 1: Arrange the coefficients of $P(x)$ and $x = 3$ in the following fashion:

$$\begin{array}{c|cccc} 3 & 1 & 1 & -10 & 8 \\ & & & & \\ \hline \end{array} \quad \leftrightsquigarrow \text{Coeff's of } P$$

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 2: Bring down the first coefficient, 1.

$$\begin{array}{c|cccc} 3 & 1 & 1 & -10 & 8 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 3: Multiply the 3 by 1, then write the result below the other 1.

3	1	1	-10	8
	1			

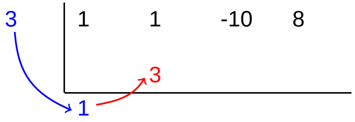
Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 3: Multiply the 3 by 1, then write the result below the other 1.

3	1	1	-10	8
	1			

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10\,x + 8$ by $x - 3$.

Step 3: Multiply the 3 by 1, then write the result below the other 1.



Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 4: Add the 1 and the 3 in the second column, writing the result below the sum.

3	1	1	-10	8
		3		
	1	4		

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 5: Multiply the 3 by 4, then write the result below −10.

3	1	1	-10	8
		3		
	1	4		

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10\,x + 8$ by $x - 3$.

Step 5: Multiply the 3 by 4, then write the result below -10.

3	1	1	-10	8
		3		
	1	4		

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 5: Multiply the 3 by 4, then write the result below -10.

3	1	1	-10	8
		3	12	
	1	4		

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 6: Add the -10 and the 12 in the third column, then write the result below the sum.

$$\begin{array}{c|cccc} 3 & 1 & 1 & -10 & 8 \\ & & 3 & 12 & \\ \hline & 1 & 4 & 2 & \end{array}$$

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 7: Multiply the 3 by 2, then write the result below the 8 in column 4.

3	1	1	-10	8
		3	12	
	1	4	2	

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 7: Multiply the 3 by 2, then write the result below the 8 in column 4.

3	1	1	-10	8
		3	12	
	1	4	2	

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 7: Multiply the 3 by 2, then write the result below the 8 in column 4.

3	1	1	-10	8
		3	12	6
	1	4	2	

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Step 7: Multiply the 3 by 2, then write the result below the 8 in column 4.

3	1	1	-10	8
		3	12	6
	1	4	2	

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

Last Step: Add the 8 and the 6 in the fourth column, then write the result below the sum.

3	1	1	-10	8
		3	12	6
	1	4	2	14

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

The bottom row of the synthetic division table gives us the coefficients of the quotient and remainder.

3	1	1	-10	8
		3	12	6
	1	4	2	14

Example: Use *synthetic division* to divide $P(x) = 1x^3 + 1x^2 - 10x + 8$ by $x - 3$.

We divided a 3rd degree polynomial by a 1st degree polynomial, so the quotient is a polynomial with degree $3 - 1 = 2$.

3	1	1	-10	8
		3	12	6
	1	4	2	14

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$1x^2 + 4x + 2 + \frac{14}{x-3}$$

The Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Example: Use the Remainder Theorem to find $P(-4)$ if $P(x) = x^3 - 8x^2 + 9x - 2$.

The Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Example: Use the Remainder Theorem to find $P(-4)$ if $P(x) = x^3 - 8x^2 + 9x - 2$.

-4	1	-8	9	-2
		-4	48	-228
	1	-12	57	-230

$$
\begin{array}{r|rrrrrrr}
 & x^2 & - & 12x & +57 & & & \\
\hline
x+4 & x^3 & - & 8x^2 & + & 9x & - & 2 \\
 & -(x^3 & + & 4x^2) & & & & \\
\hline
 & & & -12x^2 & + & 9x & - & 2 \\
 & & & -(-12x^2 & - & 48x) & & \\
\hline
 & & & & & 57x & - & 2 \\
 & & & & & -(57x & + & 228) \\
\hline
 & & & & & & & -230
\end{array}
$$

Check: $P(-4) = (-4)^3 - 8(-4)^2 + 9(-4) - 2 = -230$

The easiest way to evaluate a polynomial function is by using synthetic division, thanks to the Remainder Theorem!

Definition

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$.

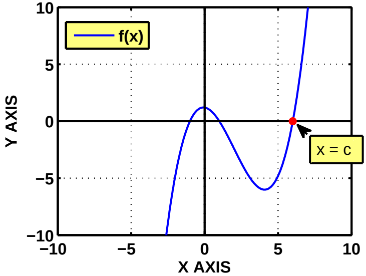


Figure : The terms *x*-intercepts, roots and zeros (of a function, *f*) have the same meaning.

Factor Theorem

$x = c$ is a **zero** of $P(x)$ if and only if $x - c$ is a **factor** of $P(x)$.

Example: Determine whether $x + 4$ is a factor of the polynomial

$$P(x) = 1x^3 + 1x^2 - 10\,x + 8$$

Factor Theorem

$x = c$ is a **zero** of $P(x)$ if and only if $x - c$ is a **factor** of $P(x)$.

Example: Determine whether $x + 4$ is a factor of the polynomial

$$P(x) = 1x^3 + 1x^2 - 10\,x + 8$$

-4	1	1	-10	8
		-4	12	-8
	1	-3	2	0

$\rightarrow$ Remainder Thm. $\Longrightarrow P(-4) = 0$

Factor Theorem

$x = c$ is a **zero** of $P(x)$ if and only if $x - c$ is a **factor** of $P(x)$.

Example: Determine whether $x + 4$ is a factor of the polynomial

$$P(x) = 1x^3 + 1x^2 - 10\,x + 8$$

-4	1	1	-10	8
		-4	12	-8
	1	-3	2	0

$\rightarrow$ Remainder Thm. $\Longrightarrow P(-4) = 0$

Hence, $x = -4$ is a zero of P. Then $x + 4$ is a factor of P by the Factor Thm. Moreover, P(x) has factorization

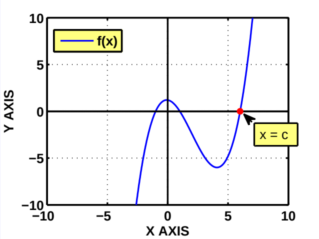
$$P(x) = (x+4)(1x^2 - 3x + 2)$$
$$= (x+4)(x-2)(x-1)$$

Also, the zeros of P are $x = -4, 2, 1$

Real Zeros

If P is a polynomial and c is a real number, then the following statements are equivalent.

1. c is a **zero** of P.
2. $x = c$ is a **solution** of the equation $P(x) = 0$.
3. $x - c$ is a **factor** of $P(x)$.
4. c is an x-**intercept** of the graph of P.



Rational Zeros Theorem

If the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then every rational zero of $P(x)$ is of the form

$$\frac{p}{q}$$

where p is a factor of the constant term a_0

q is a factor of the leading coefficient a_n

Example: Make a list of the POSSIBLE zeros of $f(x) = 2x^4 - 13x^3 + 33x^2 - 37x + 15$ using the Rational Zeros thm.

Soln: $f(x) = 2x^4 - 13x^3 + 33x^2 - 37x + 15$

factors of 15 $\quad p \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$

factors of 2 $\quad q \in \{\pm 1, \pm 2\}$

Then the RZ thm $\Longrightarrow$ the list of *POSSIBLE* rational zeros is the set

$$\frac{p}{q} \in \left\{ \pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2} \right\}$$

$\star$ the actual zeros are $x = 1, \frac{3}{2}, 2 \pm i$

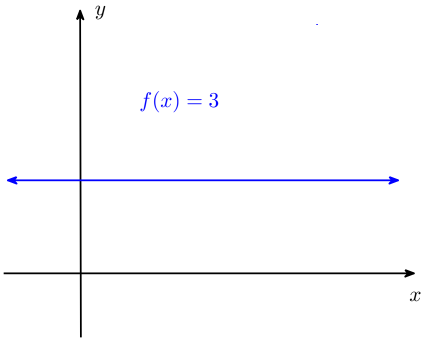
Descartes' Rule of Signs

Suppose $P(x)$ has real coefficients.

1. The number of positive real zeros of $P(x)$ either is equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.
2. The number of negative real zeros of $P(x)$ either is equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

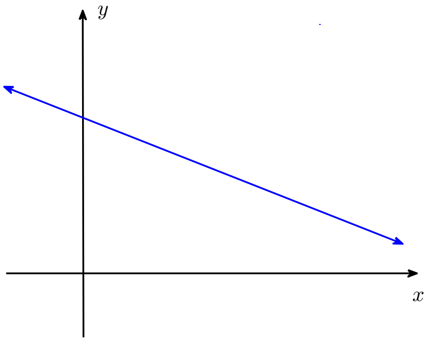
Polynomial Graphs

Graphs of polynomial functions having degree 0 or 1 are lines.



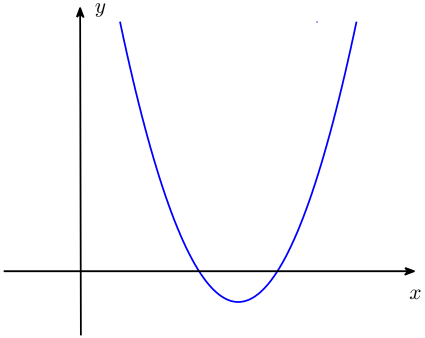
Polynomial Graphs

Graphs of polynomial functions having degree 0 or 1 are lines.



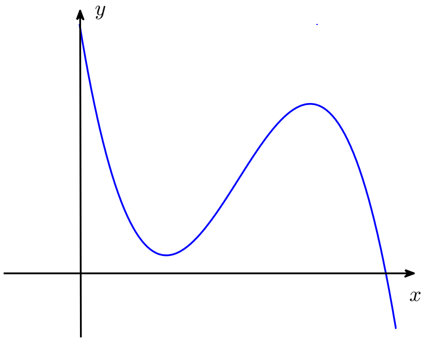
Polynomial Graphs

2nd degree polynomial graphs are parabolas.



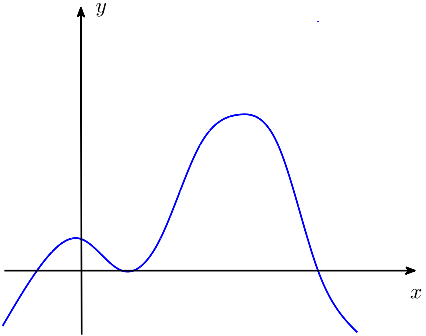
Polynomial Graphs

Graphs of polynomial functions having degree 2 or more can get more complicated.



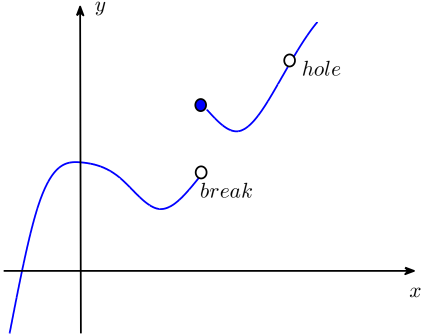
Polynomial Graphs

Graphs of polynomial functions having degree 2 or more can get more complicated.



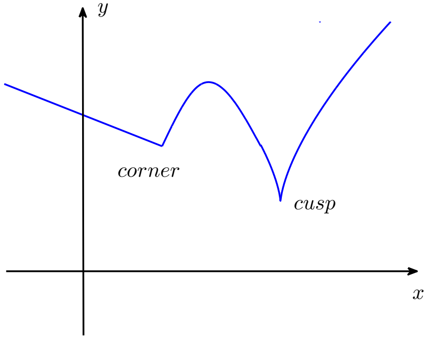
Polynomial Graphs

Yet the graph of a polynomial function is **continuous**, meaning there are no **breaks** or **holes**.



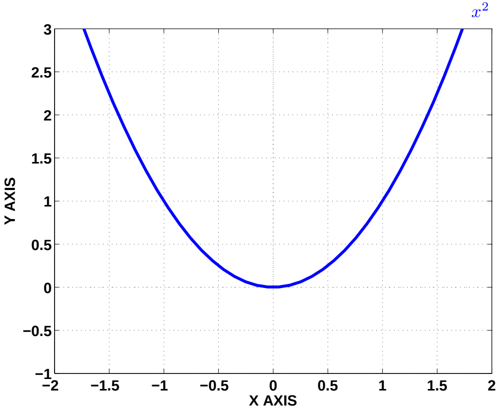
Polynomial Graphs

Moreover, the graph of a polynomial function is a **smooth** curve, meaning it has no **corners** or **cusps** (sharp points).

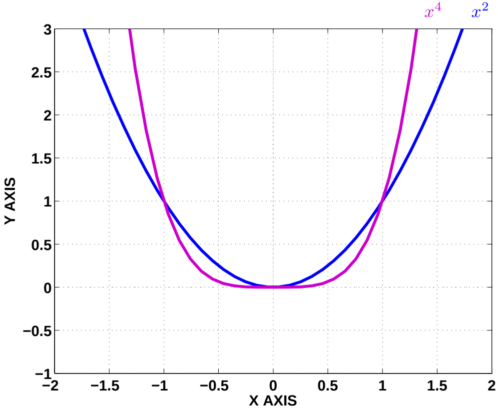


Monomial functions, $p(x) = x^n$, and their translations are the simplest polynomial functions to graph.

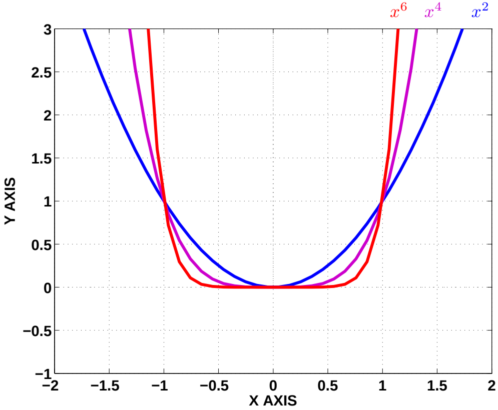
- If n is even, the graph of $p(x) = x^n$ is similar to the parabola $y = x^2$.



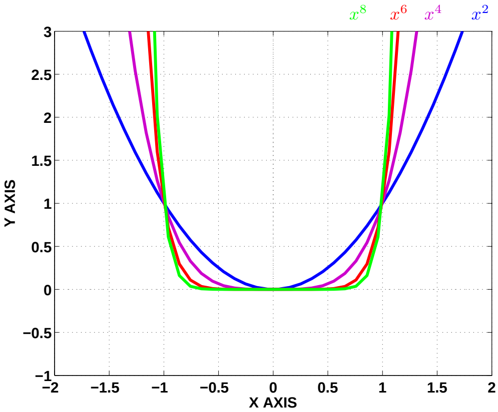
If n is even, the graph of $p(x) = x^n$ is similar to the parabola $y = x^2$.



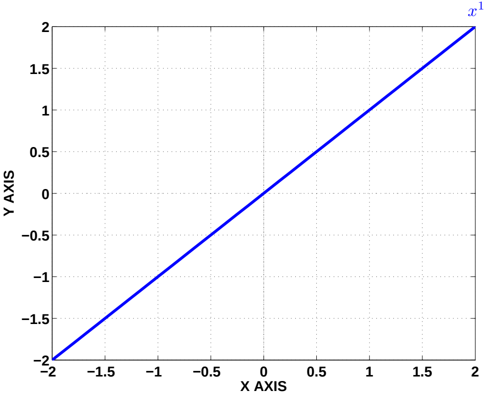
If n is even, the graph of $p(x) = x^n$ is similar to the parabola $y = x^2$.



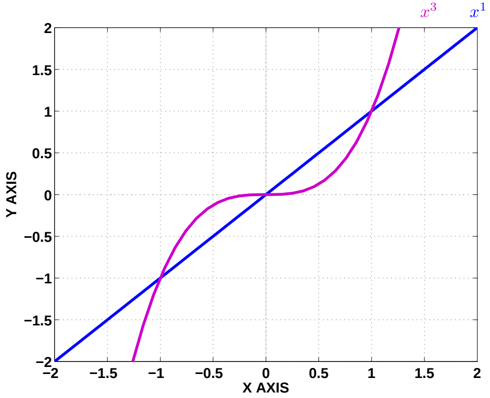
If n is even, the graph of $p(x) = x^n$ is similar to the parabola $y = x^2$.



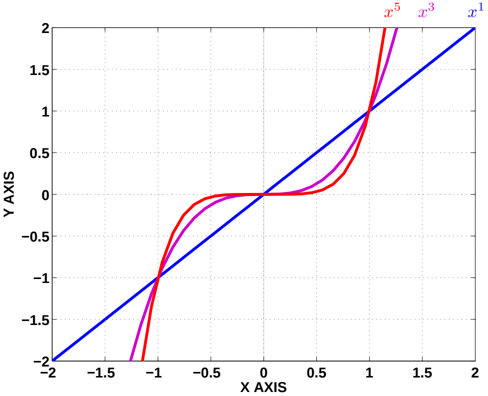
- If n is odd, the graph of $p(x) = x^n$ is similar to the cubic $y = x^3$.



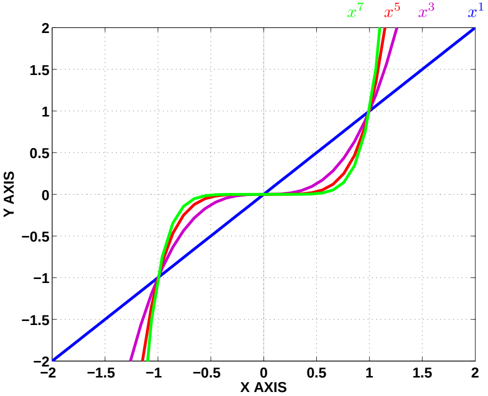
If n is odd, the graph of $p(x) = x^n$ is similar to the cubic $y = x^3$.



If n is odd, the graph of $p(x) = x^n$ is similar to the cubic $y = x^3$.



If n is odd, the graph of $p(x) = x^n$ is similar to the cubic $y = x^3$.



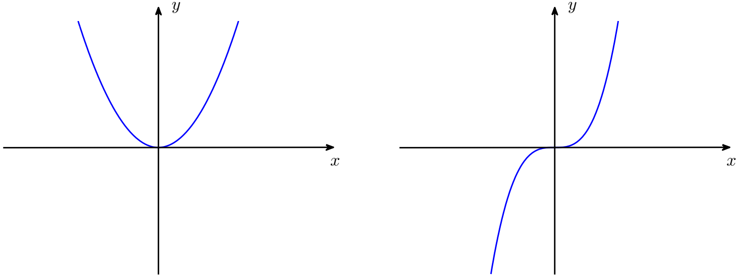
Definition

The end behavior of a polynomial is a statement describing what happens to the y value as x becomes large in the positive or negative direction.

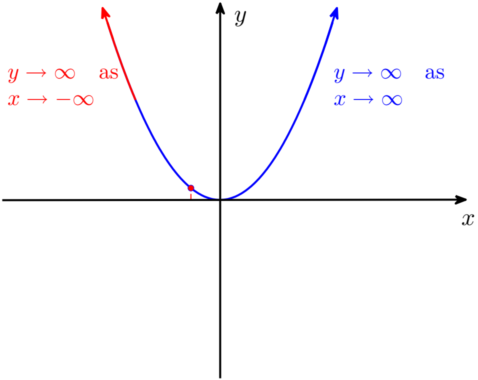
We will use this notation when making end behavior statements:

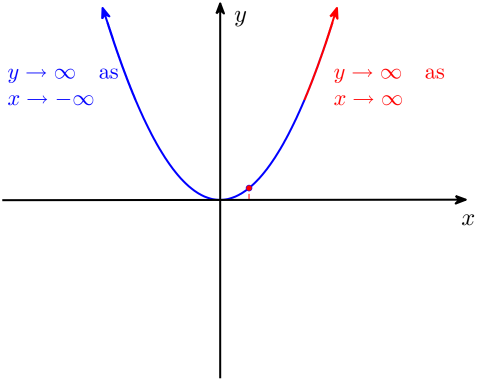
$x \to \infty$	means	"x becomes large in the positive direction"
$x \to -\infty$	means	"x becomes large in the negative direction"

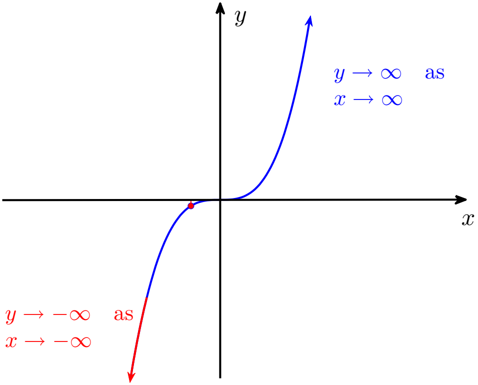
Polynomial functions have infinite end behavior: the y-value shoots off to ∞ or $-\infty$.

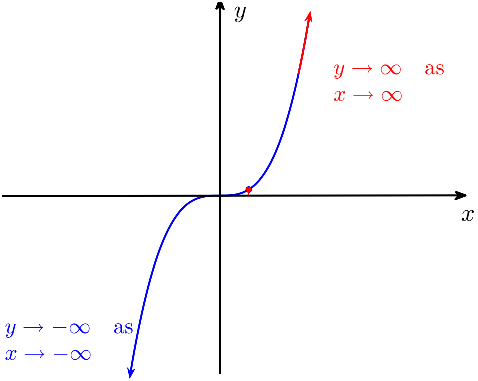


For example, consider the graphs of $y = x^2$ and $y = x^3$.


y
x
$y \to \infty$ as
$x \to -\infty$
$y \to \infty$ as
$x \to \infty$


y
$y \to \infty$ as
$x \to -\infty$
$y \to \infty$ as
$x \to \infty$
x


y
x
$y \longrightarrow \infty$ as
$x \longrightarrow \infty$
$y \longrightarrow -\infty$ as
$x \longrightarrow -\infty$


y
x
$y \to \infty$ as
$x \to \infty$
$y \to -\infty$ as
$x \to -\infty$

Dominance Property of Leading Terms

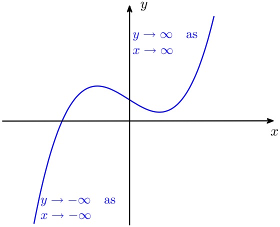
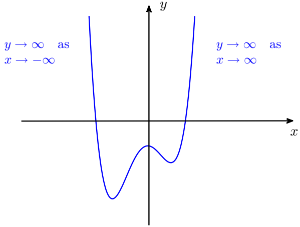
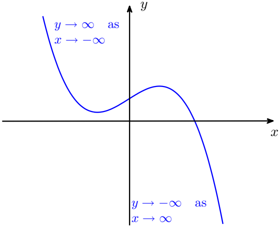
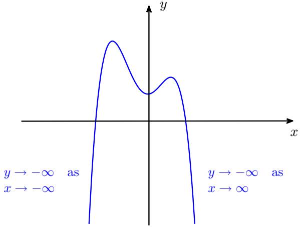
The leading term of a polynomial, $a_n x^n$, increasingly dominates the other terms and increasingly determines the shape of the graph "in the long run" (as $x \to \infty$ and as $x \to -\infty$).

The lower-degree terms work together with $a_n x^n$ to determine the shape of the graph close to the origin and near the zeros. However, in the long term the lower-degree terms become relatively insignificant in size.

The Leading Term Test

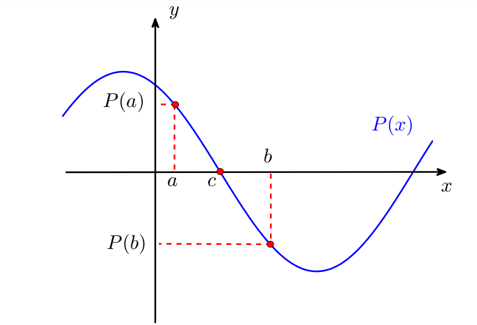
n is the degree of the polynomial

a_n is the leading coefficient

	n odd	n even
$a_n > 0$		
$a_n < 0$		

Intermediate Value Theorem (IVT)

If $P(x)$ is polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one number c between a and b for which $P(c) = 0$.



The IVT is proved in calculus. An important consequence of the theorem is that between any two successive zeros the graph of a polynomial lies *entirely above* or *entirely below* the x-axis. This validates us using the Test Point Method of solving inequalities for continuous functions.

Definition

If the factor $x - c$ occurs m times in the complete factorization of the polynomial $P(x)$, then c is called a root of $P(x) = 0$ with multiplicity m.

Definition

If the factor $x - c$ occurs m times in the complete factorization of the polynomial $P(x)$, then c is called a root of $P(x) = 0$ with multiplicity m.

Example: Determine the multiplicity of each root of $P(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$ given that P has the complete factorization

$$P(x) = x(x+2)^2(x-1)(x-2)$$

Definition

If the factor $x - c$ occurs m times in the complete factorization of the polynomial $P(x)$, then c is called a root of $P(x) = 0$ with multiplicity m.

Example: Determine the multiplicity of each root of $P(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$ given that P has the complete factorization

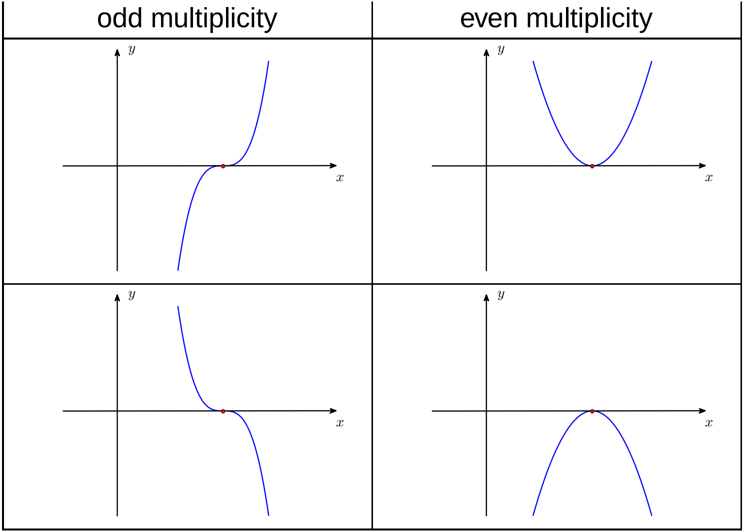
$$P(x) = x(x+2)^2(x-1)(x-2)$$

factors of $f(x)$	associated roots	multiplicity	parity
x	$x = 0$	1	odd
$(x+2)$	$x = -2$	2	even
$(x-1)$	$x = 1$	1	odd
$(x-2)$	$x = 2$	1	odd

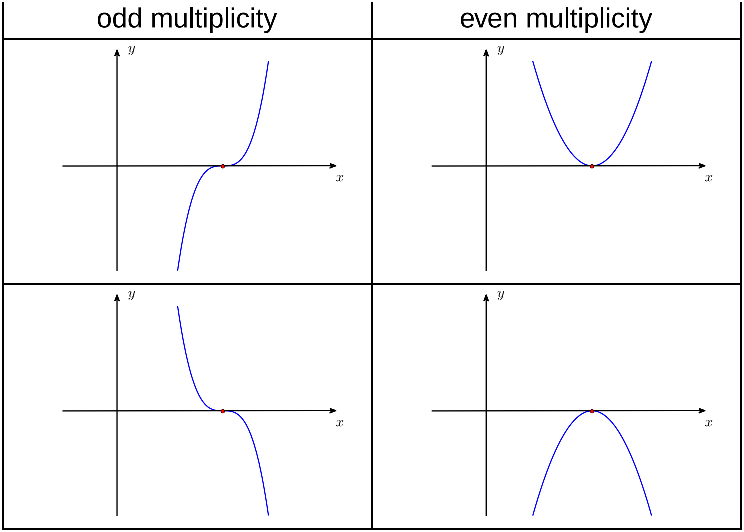
Definition

If the factor $x - c$ occurs m times in the complete factorization of the polynomial $P(x)$, then c is called a root of $P(x) = 0$ with multiplicity m.

The graph crosses the x-axis at an odd zero.



At an even zero, the graph touches, but does not cross the x-axis.



Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Soln. Factor the inequality.

$$x(x+2)^2(x-1)(x-2) > 0$$

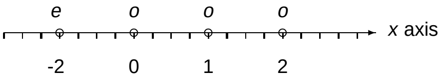
Then draw an x axis.

Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Soln. Factor the inequality.

$$x(x+2)^2(x-1)(x-2) > 0$$

Then draw an x axis. Locate and label (with a hollow circle) the real zeros of $f(x)$ along the x axis. Write an "o" or "e" above each zero, depending on whether the multiplicity of each root is even or odd.

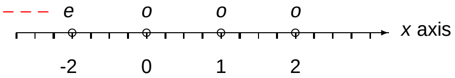


Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Soln. Now test the sign of f for x values in the *leftmost* subinterval. This is done using the end behavior of the graph of f; or by determining the functional value for a test point in $(-\infty, -2)$. For this example, we take $x = -3$ and find $f(-3)$ using synthetic division.

-3	1	1	-6	-4	8	0
		-3	6	0	12	-60
	1	-2	0	-4	20	-60

remainder thm $\Longrightarrow f(-3) = -60$

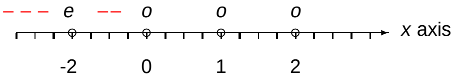


Since $f(-3) = -60 < 0$, we mark the the x axis graph with negative signs above the interval $\in (-\infty, -2)$.

Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Soln. Now mark the sign of each successive interval, from left to right using this rule:

- There *is* a sign change (in f) around the zero at $x = c$ provided that $x - c$ is a factor with odd multiplicity.
- There *is not* a sign change (in f) around the zero at $x = c$ provided that $x - c$ is a factor with even multiplicity.

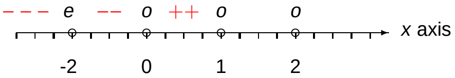


The next subinterval to test is $x \in (-2, 0)$. Since $x = -2$ has even multiplicity, *there is no sign change in f about* $x = -2$.

Hence, we mark negative signs on our x axis graph above the subinterval $(-2, 0)$ to indicate that $f(x) < 0$ for $x \in (-2, 0)$.

Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

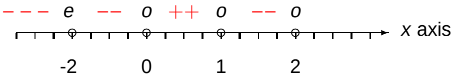
Soln. The next root, $x = 0$, has odd multiplicity, so there *is* a sign change in f about $x = 0$. Therefore, we mark our x axis graph with plus signs above the subinterval $(0, 1)$.



Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Soln. The next root, $x = 0$, has odd multiplicity, so there *is* a sign change in f about $x = 0$. Therefore, we mark our x axis graph with plus signs above the subinterval $(0, 1)$.

$x = 1$ has odd multiplicity, there is a sign change in f about $x = 1$. Therefore we mark the sign of $f(x)$ with negative signs above the subinterval $(1, 2)$ on our x axis graph.

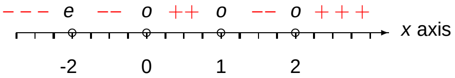


Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Soln. The next root, $x = 0$, has odd multiplicity, so there *is* a sign change in f about $x = 0$. Therefore, we mark our x axis graph with plus signs above the subinterval $(0, 1)$.

$x = 1$ has odd multiplicity, there is a sign change in f about $x = 1$. Therefore we mark the sign of $f(x)$ with negative signs above the subinterval $(1, 2)$ on our x axis graph.

Since $x = 2$ has odd multiplicity, there is a sign change in f about $x = 2$. Thus we mark positive signs above the subinterval $(2, \infty)$ on our x axis graph.

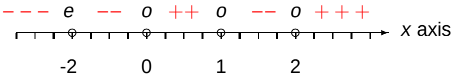


Example Solve $f(x) > 0$ if $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.

Soln. The next root, $x = 0$, has odd multiplicity, so there *is* a sign change in f about $x = 0$. Therefore, we mark our x axis graph with plus signs above the subinterval $(0, 1)$.

$x = 1$ has odd multiplicity, there is a sign change in f about $x = 1$. Therefore we mark the sign of $f(x)$ with negative signs above the subinterval $(1, 2)$ on our x axis graph.

Since $x = 2$ has odd multiplicity, there is a sign change in f about $x = 2$. Thus we mark positive signs above the subinterval $(2, \infty)$ on our x axis graph.

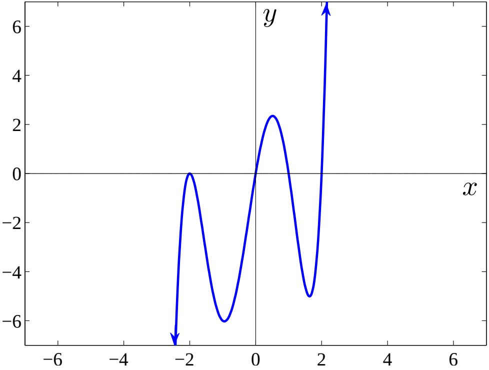


Identify the solution set: $\boxed{X \in (0, 1) \cup (2, \infty)}$

We can conclude without graphing $f(x)$ that the graph of $f(x)$ is

- *above* the x axis whenever $x \in (0, 1) \cup (2, \infty)$;
- *below* the x axis whenever $x \in (-\infty, -2) \cup (-2, 0) \cup (1, 2)$.

Example The graph of $f(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$.



We saw earlier in the chapter that if the discriminant of a quadratic equation is negative, the equation has no real solution. For example, the equation

$$x^2 + 4 = 0$$

has no real solution.

We saw earlier in the chapter that if the discriminant of a quadratic equation is negative, the equation has no real solution. For example, the equation

$$x^2 + 4 = 0$$

has no real solution. If we try to solve this equation, we get $x^2 = -4$, so

$$x = \pm\sqrt{-4}$$

We saw earlier in the chapter that if the discriminant of a quadratic equation is negative, the equation has no real solution. For example, the equation

$$x^2 + 4 = 0$$

has no real solution. If we try to solve this equation, we get $x^2 = -4$, so

$$x = \pm\sqrt{-4}$$

But this is impossible, since the square of any real number is positive. (For instance, $(-2)^2 = 4$, a positive number.) Thus, negative numbers don't have real square roots.

We saw earlier in the chapter that if the discriminant of a quadratic equation is negative, the equation has no real solution. For example, the equation

$$x^2 + 4 = 0$$

has no real solution. If we try to solve this equation, we get $x^2 = -4$, so

$$x = \pm\sqrt{-4}$$

But this is impossible, since the square of any real number is positive. (For instance, $(-2)^2 = 4$, a positive number.) Thus, negative numbers don't have real square roots.
To make it possible to solve all quadratic equations, mathematicians invented an expanded number system, called the **complex number system**. First they defined a new number,

$$i = \sqrt{-1}$$

Definition

A **complex number** is an expression of the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The **real part** of this complex number is a and the **imaginary part** is b. Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

Definition

A **complex number** is an expression of the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The **real part** of this complex number is a and the **imaginary part** is b. Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

Examples of complex numbers

$4 + 5i$	real part 4, imaginary part 4
$1 - i$	real part 1, imaginary part -1
$6i$	real part 0, imaginary part 6
-7	real part -7, imaginary part 0

Definition

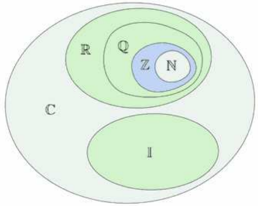
A **complex number** is an expression of the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The **real part** of this complex number is a and the **imaginary part** is b. Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

Examples of complex numbers

$4 + 5i$	real part 4, imaginary part 4
$1 - i$	real part 1, imaginary part -1
$6i$	real part 0, imaginary part 6
-7	real part -7, imaginary part 0



Definition

Addition	**Description**
$(a+bi)+(c+di)=(a+c)+(b+d)i$	To add complex numbers, add the real parts and the imaginary parts.
Subtraction	
$(a+bi)-(c+di)=(a-c)+(b-d)i$	To subtract complex numbers, subtract the real parts and the imaginary parts.
Multiplication	
$(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i$	Multiply complex numbers like binomials, using $i^2=-1$.

For a complex number $z=a+bi$, we define its **complex conjugate** to be $\bar{z}=a-bi$. Note that

$$z\cdot\bar{z}=(a+bi)(a-bi)=a^2+b^2$$

Dividing Complex Numbers

To simplify the quotient $\dfrac{a+bi}{c+di}$, multiply by 1 in the form of the denominator's conjugate divided by itself.

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Square Roots of Negative Numbers

If $-r$ is negative, then the positive (principal) square root of $-r$ is

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of $-r$ are $i\sqrt{r}$ and $-i\sqrt{r}$

Dividing Complex Numbers

To simplify the quotient $\dfrac{a+bi}{c+di}$, multiply by 1 in the form of the denominator's conjugate divided by itself.

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Square Roots of Negative Numbers

If $-r$ is negative, then the positive (principal) square root of $-r$ is

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of $-r$ are $i\sqrt{r}$ and $-i\sqrt{r}$

We usually write $i\sqrt{b}$ instead of $\sqrt{b}i$ to avoid confusion with $\sqrt{bi}$.

CAUTION

$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ when both a and b are positive, this is not true when both a and b are negative. For instance

$$\sqrt{-2} \cdot \sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}$$

but

$$\sqrt{(-2) \cdot (-3)} = \sqrt{6}$$

so

$$\sqrt{-2} \cdot \sqrt{-3} \neq \sqrt{(-2) \cdot (-3)}$$

We have already seen that if $a \neq 0$, the solutions of the quadratic equation $ax^2 + bx + c$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, then the equation has no real solution. But in the complex number system, this equation will always have solutions, because negative numbers have square roots in this expanded setting.

1. ***Fundamental Theorem of Algebra***: A polynomial function (or polynomial equation) with real coefficients of degree n has at most n distinct real roots, and exactly n (not necessarily distinct) complex roots.

2. ***Corollary***: If the polynomial P has real coefficients and if the complex number z is a zero of P, then its complex conjugate $\bar{z}$ is also a zero of P.

3. ***Corollary***: An n^{th} degree polynomial has n roots and can be factored into a product of n linear factors; i.e.,
$P(x) = (a_1x + b_1) \cdot (a_2x + b_2) \cdot (a_3x + b_3) \cdots (a_{n-1}x + b_{n-1}) \cdot (a_nx + b_n)$

4. ***Corollary***: A polynomial of even degree has a minimum of zero real roots. A polynomial of odd degree has at least one real root.

5. ***Clarification***: The solutions (roots) of a polynomial equation $P(x) = 0$ are precisely the zeros of a polynomial function $y = P(x)$. Hence, the aforementioned theorems concerning zeros of polynomial functions also apply to roots of polynomial equations.

Definition

A Rational Function is a function that has the form:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The domain of f is the set of all real numbers x, such that $q(x) \neq 0$.

Definition

A Rational Function is a function that has the form:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The domain of f is the set of all real numbers x, such that $q(x) \neq 0$.

Examples:

$$f(x) = \frac{1}{x} \qquad f(x) = \frac{2-x}{3x+1} \qquad f(x) = \frac{4x-3}{x^2+3x-4}$$

Is $f(x)\dfrac{x^3-x-3}{x^{1/2}+3}$ a rational function?

Definition

A Rational Function is a function that has the form:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The domain of f is the set of all real numbers x, such that $q(x) \neq 0$.

Finding Intercepts

- Set $p(x) = 0$ and solve for x to find the x intercept(s).
- Evaluate $f(0)$ to find the y intercept.

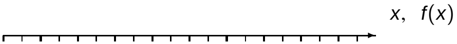
Example Solve $f(x) > 0$ if $f(x) = \dfrac{x^2 + 2x - 8}{x^3 + 2x^2}$

Example Solve $f(x) > 0$ if $f(x) = \dfrac{x^2 + 2x - 8}{x^3 + 2x^2}$

Solution: Factor numerator and denominator

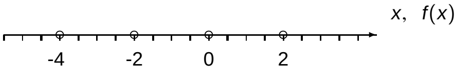
$$f(x) = \frac{(x+4)(x-2)}{x^2(x+2)}$$

Then draw an x axis.



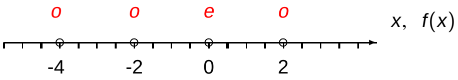
Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

Solution: Locate and label the zeros of both the numerator and denominator polynomials along the x-axis.



Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

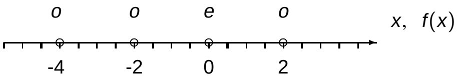
Solution: Locate and label the zeros of both the numerator and denominator polynomials along the x-axis. Place an "o" or "e" above each root to indicate whether each root is even or odd.



Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

Solution: Test the sign of the function in the leftmost subinterval, $(-\infty, -4)$.

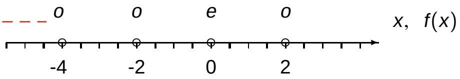
$$f(-5) = \frac{(-5+4)(-5-2)}{(-5)^2(-5+2)} = \frac{(-)(-)}{(+)(-)} < 0$$



Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

Solution: Test the sign of the function in the leftmost subinterval, $(-\infty, -4)$.

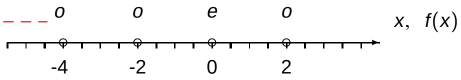
$$f(-5) = \frac{(-5+4)(-5-2)}{(-5)^2(-5+2)} = \frac{(-)(-)}{(+)(-)} < 0$$



Since the test value $x = -5$ produced $f(x) < 0$, we mark minus signs above our x axis.

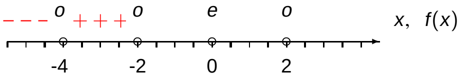
Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

Solution: Since $x = -4$ is an odd zero, the is a change in the sign of y around $x = -4$.



Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

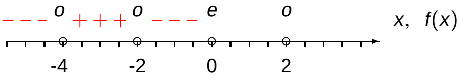
Solution: Since $x = -4$ is an odd zero, the is a change in the sign of y around $x = -4$.



Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

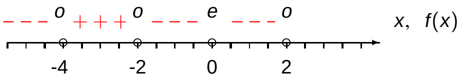
Solution: Since $x = -4$ is an odd zero, the is a change in the sign of y around $x = -4$.

$x = -2$ is an odd zero, the is a change in the sign of y around $x = -2$



Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

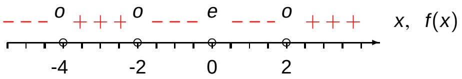
Solution:



Example Solve $f(x) > 0$ if $f(x) = \dfrac{(x+4)(x-2)}{x^2(x+2)}$

Solution:

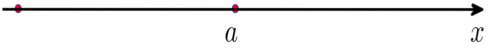
$$\boxed{x \in (-4,-2) \bigcup (2, \infty)}$$



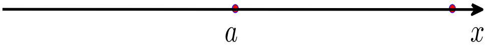
Before we graph a rational function, it is important for us to determine

1. the behavior of the functional values for x values close to a zero of the denominator;
2. end behavior: a description of $f(x)$ as x is large positive or large negative.

Notation	Terminology
$x \to a^-$	x approaches a from the left (through values *less* than a).



Notation	Terminology
$x \to a^-$	x approaches a from the left (through values *less* than a).
$x \to a^+$	x approaches a from the right (through values *greater* than a).



Notation	Terminology
$x \to a^-$	x approaches a from the left (through values *less* than a).
$x \to a^+$	x approaches a from the right (through values *greater* than a).
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large as desired).
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired).

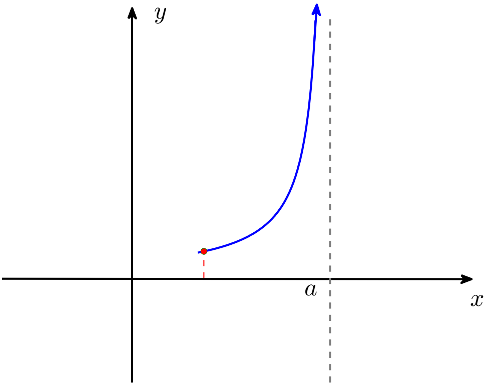
This is true unless f is like $f(x) = \dfrac{x^2-4}{x+2}$, where there is just a hole in the graph at $(x, y) = (-2, -4)$

Notation	Terminology
$x \to a^-$	x approaches a from the left (through values *less* than a).
$x \to a^+$	x approaches a from the right (through values *greater* than a).
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large as desired).
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired).

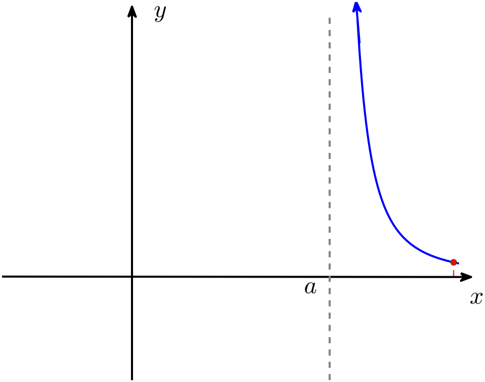
If $x = a$ is a zero of the denominator, $f(x)$ grows without bound as x approaches a.

This is true unless f is like $f(x) = \dfrac{x^2-4}{x+2}$, where there is just a hole in the graph at $(x, y) = (-2, -4)$

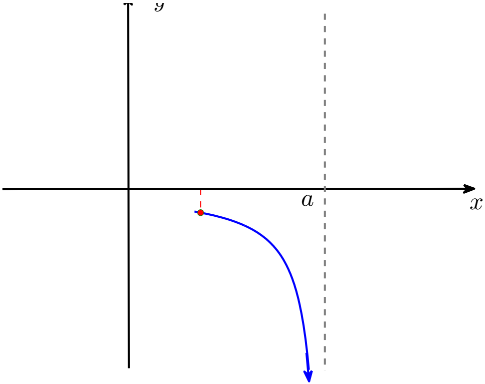
$f(x) \to \infty$ as $x \to a^-$



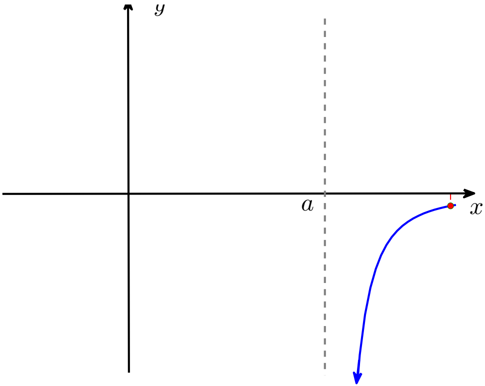
$f(x) \to \infty$ as $x \to a^+$



$f(x) \to -\infty$ as $x \to a^-$



$f(x) \to -\infty$ as $x \to a^+$



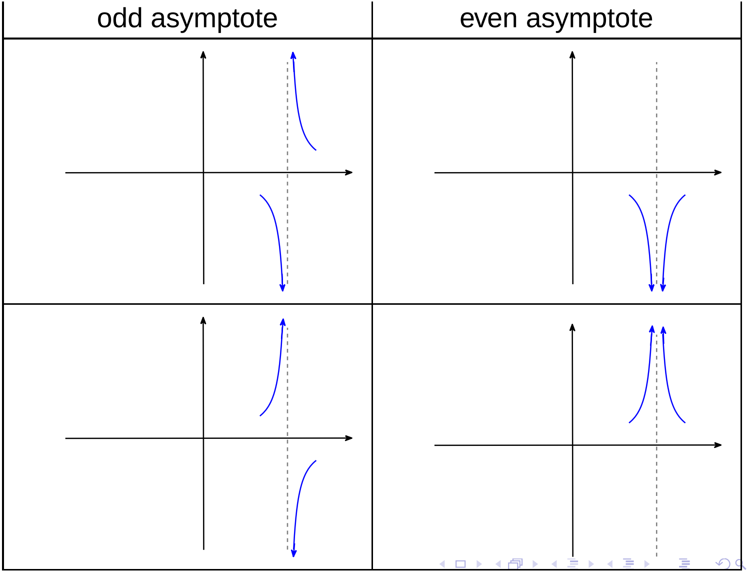
Definition

The line $x = a$ is a vertical asymptote (VA) for the graph of a function f if

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

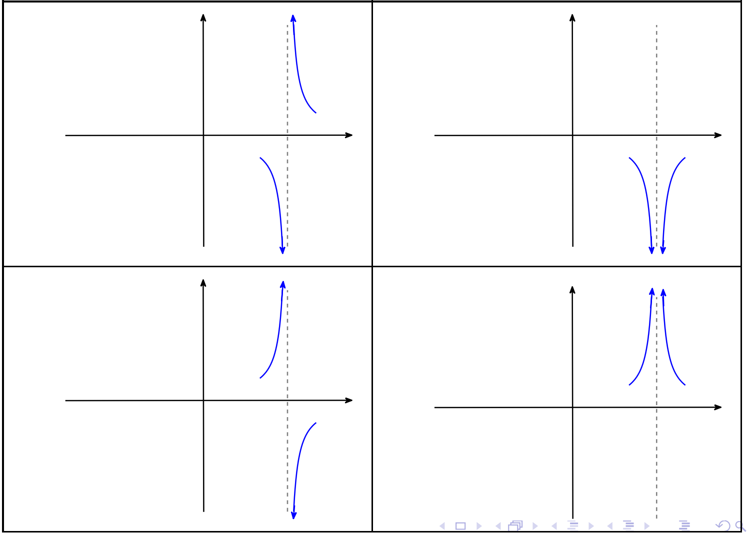
as x approaches a from either the left or the right.

The graph tends to opposite infinities around an odd vertical asymptote.



The graph tends to the same infinity around an even vertical asymptote.

odd asymptote	even asymptote

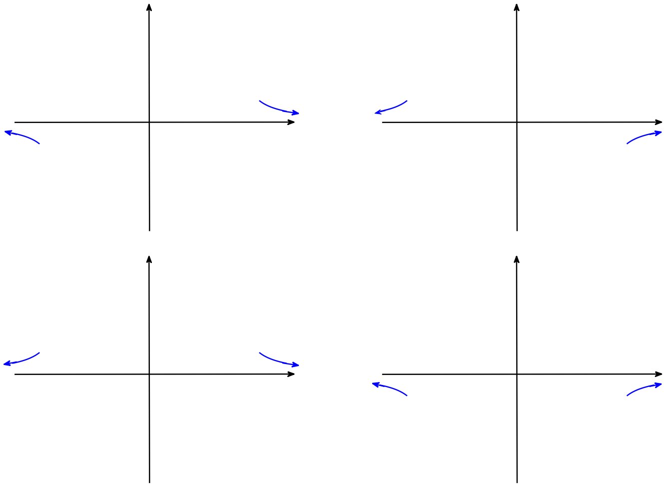


Definition

The line $y = c$ is a horizontal asymptote (HA) for the graph of a function f if

$$f(x) \to c \quad \text{as} \quad x \to \infty \quad \text{or} \quad x \to -\infty$$

Horizontal Asymptotes



Theorem

Let $f(x) = \dfrac{ax^n + ...}{bx^d + ...}$, where ax^n and bx^d are the leading terms of the polynomial in the numerator and denominator. Note that n is the degree of the numerator polynomial and d is the degree of the denominator polynomial.

1. if $n < d$, then the HA occurs at $y = 0$.
2. if $n = d$, then the HA occurs at $y =$ the ratio of the leading coefficients.
3. if $n > d$, then the graph of $f(x)$ has a slant asymptote; found by using long division to divide the polynomial in the numerator by the polynomial in the denominator. y equals the quotient is the equation representing the slant asymptote.

End Behavior As $x \to \infty$ and as $x \to -\infty$, $f(x) \to c$, where $y = c$ is the HA. If the graph has a slant asymptote, then the end behavior of f is the same as the end behavior of the the polynomial quotient after doing the long division.

The graph of a rational function will never cross a VA or SA; but it can cross a HA.